MATHEMATICS LEARNING MODULE VIII
APPLICATIONS OF INTEGRATION:
A DIFFERENTIAL APPROACH

by

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HOW TO USE THIS MODULE

Everyone should read the sections on AREAS and VOLUMES. These topics are basic. Work out the problems yourself as you follow them in the text. Fill in the missing steps. After you finish each section, turn to the book used in your course and start doing homework problems. The only way you can really learn this material is to SOLVE LOTS OF PROBLEMS. Refer back to the module as needed. See your tutor or TA if you get stuck.

Your course will require you to master some or all of the remaining topics in the module. The basic formulas are given in each section along with a common sense derivation. It's a lot easier to remember the formulas if you have an intuitive understanding of why they are true. Also, you'll make fewer mistakes in applying them. The module provides easy reading for understanding them. The textbook in the course provides a more detailed discussion. If you're having trouble reading the book, read the module first, then tackle the book.

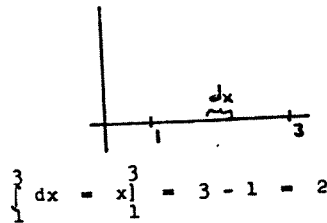The module should provide a quick review for studying for exams.

CONTENTS

INTRODUCTION

The integral sign $\int$ stands for the S in the word "summation." Integrals add things up. We will use an "intuitive" approach, using differentials (dx and dy). Think of dx and dy as very short distances in the x and y directions, respectively. So, if I add up all the dx's between x=1 and x=3, I should get the distance from 1 to 3:
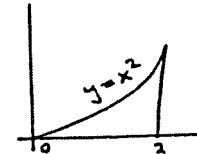

$$\int_1^3 dx = x\Big]_1^3 = 3 - 1 = 2$$

Likewise,

$$\int_{-2}^3 dy = y\Big]_{-2}^3 = 3 - (-2) = 5, \text{ which is the total}$$

distance from y=-2 to y=3.

(WARNING: Remember, $\int dx$ is really a limit, and if you want to be absolutely sure that differentials are giving you the right answer, you have to prove it with limits in a given problem. In fact, sometimes, in complex problems, differentials will give you the wrong answer if you happen to view the problem wrong. But this won't happen in freshman calculus as long as you stick to the methods discussed in this module and in your book. So now, ignore the warning, because scientists and engineers use differentials all the time to get the right
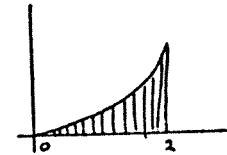
answers. They are easy to use and visualize. Just remember that, if you get around to taking any "higher" mathematics, there is more to it which you can learn about then.)
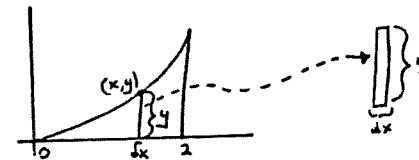
I. AREA UNDER A CURVE

Suppose I want to find the area shown below:

We are going to use an integral to add up the "areas" of all the vertical lines shown:
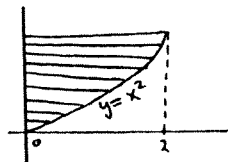

Look at a typical vertical line:

We are going to view it as a "rectangle" with base of length dx and height of length y. So, the area of the rectangle is (height)(base) = ydx. I want to add up the "areas"
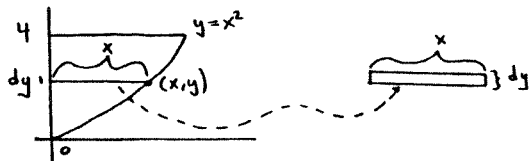
of all the lines between x=0 and x=2, so:

$$A = \int_0^2 y\, dx = \int_0^2 x^2\, dx = \left.\frac{x^3}{3}\right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$$

Suppose I want the area below instead:

Then I might want to add up all the horizontal lines shown between y=0 and y = ?. Use $y = x^2$ at x=2 to find y=4. Look at a typical line:
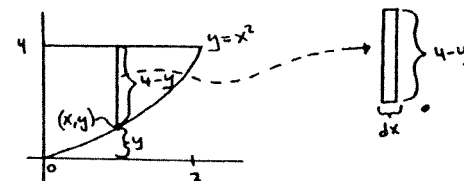

Its "area" is xdy.

So,

$$A = \int_0^4 x\, dy = \int_0^4 \sqrt{y}\, dy = \left.\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^4 = \frac{2}{3}(8) - 0 = \frac{16}{3}$$

Notice you have to solve for x in terms of y in the original equation so there will be only y's in the integral.

That is: $y = x^2$. Solve for x: get $x = \sqrt{y}$.

We could have solved the last problem using <u>vertical</u> lines instead:

$$A = \int_0^2 (4-y)\, dx = \int_0^2 (4-x^2)\, dx = \left.4x - \frac{x^3}{3}\right|_0^2$$

$$= (8 - \frac{8}{3}) - (0 - 0) = \frac{16}{3}$$

Also, we could have done the first problem with <u>horizontal</u> lines:

$$A = \int_0^4 (2-x)\, dy = \int_0^4 (2 - \sqrt{y})\, dy = \left.2y - \frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^4$$

$$= 8 - \frac{2}{3}(8) = \frac{8}{3}.$$

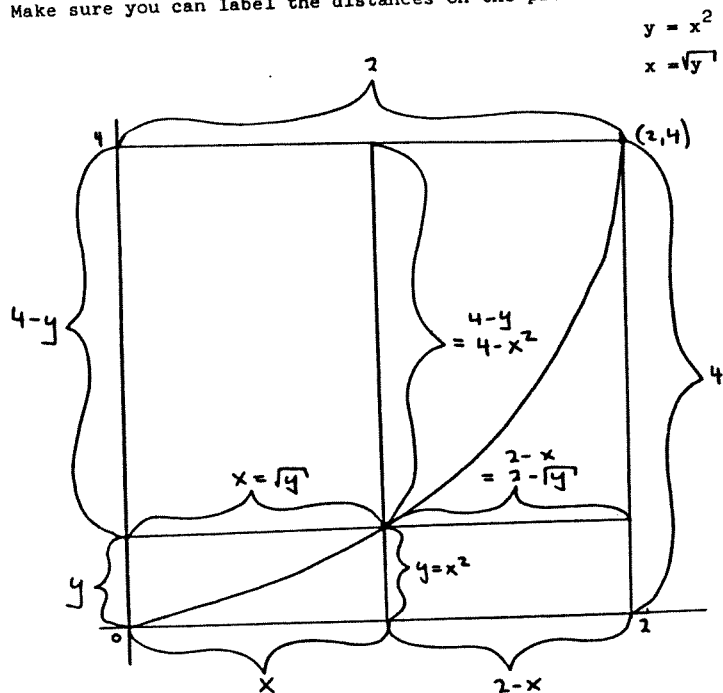Remember, it is important to be able to exploit the geometry of the pictures you draw. For instance, a typical point (x,y) on the curve $y=x^2$ between x=0 and x=2 gives rise to many horizontal and vertical distances which you can calculate in terms of x and y and use in dx and dy integrals. Make sure you can label the distances on the picture below:

$y = x^2$

$x = \sqrt{y}$

So how do you know which lines to use: horizontal or vertical. You will gradually develop a feeling for this as you do more and more practice problems. Sometimes one way is easier; sometimes the other way is impossible.

For instance: Find the area trapped between x = 0 and $x = y^8 - 2y^7$.

You have x in terms of y and you don't want to think about trying to get y in terms of x. So, you're going to use distances measured in x's, that is, horizontal lines.

First, DRAW A PICTURE. Getting a good picture is half the battle and will prevent 90 per cent of the mistakes.

Set x = 0 to find the roots along the y-axis:

$$x = y^8 - 2y^7 = 0$$
$$y^7(y - 2) = 0$$
$$y = 0 \text{ and } y = 2$$

Since $y^8 > 2y^7$ for large $|y|$, there will be large positive x's on the graph, but not large negative x's. So you know the graph "opens" to the right. This is enough for a rough sketch:

The "area" of the typical line is seen to be xdy, so

$$A = \int_0^2 x\,dy = \int_0^2 (y^8 - 2y^7)dy = \frac{y^9}{9} - \frac{2y^8}{8} \Big|_0^2 = \frac{2^9}{9} - \frac{2^8}{4}$$

The answer, by the way, is negative, since all the x-distances we added up were negative. If you want a positive area, then you have to take the absolute value of your answer.

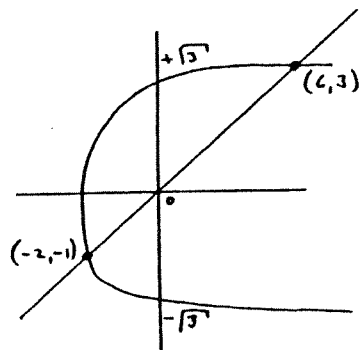Here's a typical exam problem: Find the area bounded by $x - y^2 + 3 = 0$ and $x - 2y = 0$.
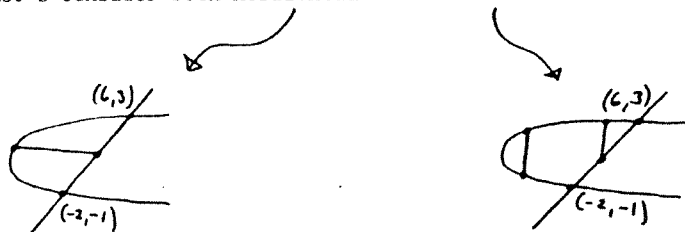
<u>DRAW</u> <u>FIRST</u>:

Here is the reasoning behind the drawing. $x = y^2 - 3$ is a parabola opening right with roots: $x = y^2 - 3 = 0$

$$(y+\sqrt{3})(y-\sqrt{3}) = 0$$
$$y=-\sqrt{3} \text{ and } y=\sqrt{3} \quad .$$

$y = \dfrac{x}{2}$ is a straight line through the origin with positive slope. Set the two curves equal and calculate the intersection points.
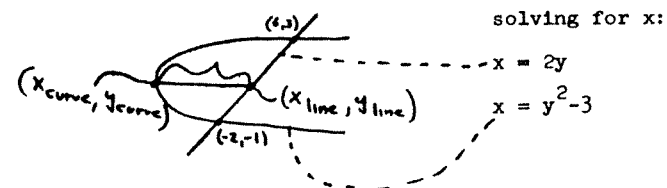
Let's consider both horizontal and vertical lines:

Notice that the <u>vertical</u> lines are of two kinds. The ones on the left of the picture connect curve to curve, whereas the ones on the right connect curve to line. The <u>horizontal</u> lines are of only one kind, so we chose them for simplicity.

What is the length of the typical horizontal line?

Horizontal length = (right minus left) = (line - curve) = $2y - (y^2 - 3)$, as can be seen from the picture below:

solving for x:
$$x = 2y$$
$$x = y^2 - 3$$

Thus, using an integral to add up all the lines between $y = -1$ and $y = 3$, we have

$$A = \int_{-1}^{3} 2y - (y^2-3)\,dy = \frac{2y^2}{2} - \frac{y^3}{3} + 3y \Big|_{-1}^{3}$$

$$= (3^2 - \frac{3^3}{3} + 3(3)) - (1 + \frac{1}{3} - 3) = 10\frac{2}{3}$$

(You might try to work this problem using vertical lines. It <u>can</u> be done.)

Remember to THINK before you dive into any solution. For instance, you can save a lot of time by noticing the geometry of the following two problems:

1) Evaluate $\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sin x \, dx$. If you draw the curve you can see that the answer is zero by symmetry. Areas above the x-axis cancel out areas below the x-axis.

2) Evaluate $\int_{-4}^{+4} \sqrt{16-x^2} \, dx$. This is just a semicircle of radius 4. So $A = \dfrac{\pi(4)^2}{2}$.

II. VOLUMES

Now let's do some volumes. Suppose our curve is revolved around the x-axis to give the volume shown below:

Each vertical line sweeps out a disc.

This disc has radius = y and height = dx, as shown.

Volume of this disc = $\pi(\text{radius})^2(\text{height}) = \pi y^2 dx$.

Adding up the volumes of all the discs for all the vertical lines between x=0 and x=2 gives:

$$V = \int_0^2 \pi y^2 dx = \int_0^2 \pi(x^2)^2 dx = \frac{\pi x^5}{5}\Big|_0^2 = \frac{32\pi}{5}$$

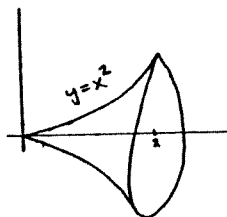There is another way to get this volume, by noticing that the horizontal lines sweep out a shell, (a hollow cylinder). The great number of shells produced by all the horizontal lines then fit inside each other (somewhat like Chinese boxes) to produce the volume.

To figure out the "volume" of the shell, think of it as a tin can only dy thick with no top or bottom. Cut it and unroll it as shown.

Now, it's an ordinary parallelepiped with volume = (length)(width)(height) = 2π(radius)(width)(height) = 2πy(2-x)dy.

$$V = \int_0^4 2\pi y(2-x)dy$$
$$= \int_0^4 2\pi y(2-\sqrt{y})dy \quad \text{[Fill in missing steps]}$$

$$= \frac{32\pi}{5}, \text{ the same answer as before}$$

For revolutions around the x-axis, we can summarize our

volume formulas:

DISCS

$$V = \int \pi(\text{radius})^2 dx$$

SHELLS

$$V = \int 2\pi(\text{radius})(\text{height}) dy$$

Which is easier? Discs or shells? It depends on the problem.
Shells may have seemed more complicated than discs in the
problem we just did, but let's try something different:

We continue to look at our same area:

But now we revolve it around the y-axis.

A typical vertical line (below) revolves to form a shell
with the dimensions shown:

$$V = \int_0^2 2\pi(\text{radius})(\text{height}) dx = \int_0^2 2\pi xy\, dx$$

$$= 2\pi \int_0^2 x(x^2) dx = \frac{2\pi x^4}{4}\Big|_0^2 = \frac{2\pi(16)}{4} = 8\pi$$

So, shells seem easier when revolving around the y-axis.
But remember our previous problem involving $x = y^8 - 2y^7$, in
which y was not available in terms of x. There, in order to
revolve around the y-axis, we could not have used shells
because there would be no way to substitute the height into
the dx integral. Discs would have to be used with radius x
in a dy integral:

We can also do the problem below, which we've already done with shells, with discs, except I have to subtract small discs from big discs to get rid of the hole in the volume:

(A disc with a hole in it is sometimes called a washer.)

$$V = \int_0^4 \pi(\text{big radius})^2 dy \;-\; \int_0^4 \pi(\text{small radius})^2 dy$$

$$= \int_0^4 \pi(2)^2 dy \;-\; \int_0^4 \pi(\sqrt{y})^2 dy$$

$$= \pi(4)y\Big|_0^4 \;-\; \frac{\pi y^2}{2}\Big|_0^4 \;=\; 16\pi - \frac{\pi(16)}{2} \;=\; 8\pi, \text{ the same answer}$$

We can now summarize our volume formulas for revolutions around the y-axis:

DISCS

$$V = \int \pi(\text{radius})^2 dy$$

SHELLS

$$V = \int 2\pi(\text{radius})(\text{height}) dx$$

We will now revolve the following area around several different axes:

PROBLEM: Revolve around the line y = -1.

DISCS

$$V = \int_1^2 \pi(\text{big radius})^2 dx \;-\; \int_1^2 \pi(\text{small radius})^2 dx$$

$$= \int_1^2 \pi(1+y)^2 dx \;-\; \int_1^2 \pi(2)^2 dx$$

$$= \int_1^2 \pi(1+x^2)^2 dx \;-\; \int_1^2 \pi(4) dx$$

[From now on, we will only set up the integrals.]

SHELLS

$$V = \int_1^4 2\pi(\text{radius})(\text{height})dy = \int_1^4 2\pi(y+1)(2-x)dy$$

$$= \int_1^4 2\pi(y+1)(2-\sqrt{y})dy$$

PROBLEM: Revolve the same area around the line x = 5. (Now you draw.)

$$\text{ans} = \int_1^4 \pi(5-\sqrt{y})^2 dy - \int_1^4 \pi(3)^2 dy \quad \text{DISCS}$$

$$= \int_1^2 2\pi(5-x)(x^2-1)dx \quad \text{SHELLS}$$

PROBLEM: Revolve the same area around the line y = 6.

$$\text{ans} = \int_1^2 \pi(5)^2 dx - \int_1^2 \pi(6-x^2)dx \quad \text{DISCS}$$

$$= \int_1^4 2\pi(6-y)(2-\sqrt{y})dy \quad \text{SHELLS}$$

MORE VOLUME PROBLEMS

Suppose the area shown below is the <u>base</u> of a solid:

Suppose that, when you slice this solid with a plane perpendicular to the x-axis, you get a square of side y:

The volume of each slice = (length)(width)(height) = $y^2 dx$. If we add up all the slices between x=0 and x=2, we get the total volume of the solid:

$$V = \int_0^2 y^2 dx = \int_0^2 (x^2)^2 dx = \frac{x^5}{5}\Big|_0^2 = \frac{32}{5}$$

Suppose each slice was a semicircle:

$$V = \int_0^2 \frac{\pi(\text{radius})^2}{2} dx = \int_0^2 \frac{\pi\left(\frac{y}{2}\right)^2}{2} dx = \int_0^2 \frac{\pi y^2}{8} = \frac{\pi}{3}$$

The slice might be any shape for which the area can be calculated as a function of y. Slices taken perpendicular to the y-axis can also be used to define the solid, in which case the cross-sectional area would be a function of x and you would use a dy integral to calculate the volume.

III. ARC LENGTH

$y = x^{\frac{3}{2}}$

Suppose you want to calculate the length of the above curve. Then $L = \int_{1}^{3} ds$, where ds is "a short distance along the curve," which we shall calculate in terms of dx and dy:

By Pythagoras $dx^2 + dy^2 = ds^2$

$$ds = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{\frac{(dx^2 + dy^2)(dx^2)}{dx^2}}$$

[multiply top and bottom by dx^2]

$$ds = \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

Our problem is now solved by adding up all the ds's between x=1 and x=3: ($y = x^{\frac{3}{2}}$, so $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$)

$$L = \int_{1}^{3} ds = \int_{1}^{3}\sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

$$= \int_{1}^{3}\sqrt{1 + \frac{9x}{4}}\, dx$$

The formula looks a little different if we do the same problem with a dy integral:

$y = x^{\frac{3}{2}}$

$$ds = \sqrt{1 + (\frac{dx}{dy})^2}\, dy$$

[multiply top and bottom by dy^2 instead of dx^2]

Now, since $y = x^{\frac{3}{2}}$, $x = y^{\frac{2}{3}}$, $\frac{dx}{dy} = \frac{2}{3}y^{-\frac{1}{3}}$,

$$L = \int_{1}^{3^{\frac{3}{2}}}\sqrt{1 + \frac{4}{9}y^{-\frac{2}{3}}}\, dy$$

(You haven't been told how to work out this integral yet.)

[Helpful Hint: $\int\sqrt{1 + (\text{mess})^2}$ is often hard to evaluate in arc length problems. So, the $(\text{mess})^2$ is often rigged to simplify the arithmetic on exams. Suppose $(\text{mess})^2 = (x^2 - \frac{1}{4x^2})^2$, for instance. Then $1 + (\text{mess})^2$

$= 1 + (x^4 - \frac{1}{2} + \frac{1}{16x^4}) = x^4 + \frac{1}{2} + \frac{1}{16x^4} = (x^2 + \frac{1}{4x^2})^2$.

Now, $\sqrt{1 + (\text{mess})^2} = x^2 + \frac{1}{4x^2}$. The root is gone and the integral is easy.]

If the curve is parametrized (For example, our $y=x^{\frac{3}{2}}$ can be parametrized $x=t^2$, $y=t^3$ as t goes from 1 to $\sqrt{3}$), then there is yet another formula for ds:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

In our problem, $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2$, so:

$$L = \int_1^{\sqrt{3}} \sqrt{4t^2 + 9t^4}\ dt$$

EXAM PROBLEM: Derive the formula for the circumference of a circle of radius R.

Parametrize the circle in terms of α:

$$x = R\cos\alpha \quad \frac{dx}{d\alpha} = -R\sin\alpha$$

$$y = R\sin\alpha \quad \frac{dy}{d\alpha} = R\cos\alpha$$

$$0 \le \alpha \le 2\pi$$

$$L = \int_0^{2\pi} \sqrt{(-R\sin\alpha)^2 + (R\cos\alpha)^2}\ d\alpha$$

$$= \int_0^{2\pi} \sqrt{R^2(\sin^2\alpha + \cos^2\alpha)}\ d\alpha$$

$$= \int_0^{2\pi} R\ d\alpha = R\alpha\Big|_0^{2\pi} = 2\pi R$$

Let's revolve the curve $y=x^2$ around the x-axis. This time we ask for the surface area of the object obtained. Notice that the object is hollow in this case.

Consider what happens when a typical ds is revolved:

The surface area of the frustrum obtained = $2\pi(\text{radius})(\text{slant height}) = 2\pi(\text{radius})ds$. Here, the radius is y. Using the integral to add up all the frustrums for all the ds's between x=0 and x=2, we have the total surface area:

$$SA = \int_0^2 2\pi y\,ds \qquad\qquad y = x^2$$

$$= \int_0^2 2\pi x^2 \sqrt{1 + (2x)^2}\ dx \qquad \frac{dy}{dx} = 2x$$

The general formula is $\boxed{SA = \int 2\pi(\text{radius})ds}$

Now, revolve the same curve around the **y-axis**:

$$SA = \int 2\pi(\text{radius})ds = \int_0^4 2\pi(x)ds$$

$$= \int_0^4 2\pi(\sqrt{y})\sqrt{1 + \frac{1}{4y}}\ dy$$

(Remember that the ds in a dy integral involves a $\left(\frac{dx}{dy}\right)^2$. Since $x = \sqrt{y}$, $\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$.)

V. HYDROSTATIC PRESSURE

A fluid exerts a pressure at a depth h equal to (density)(depth) = ρh, where ρ is the density of the fluid.

Force = (pressure)(area). So, for instance, the force on the **bottom** of the can full of water shown below = (pressure)(area) = (density)(depth)$(\pi)(\text{radius})^2$ = $w(5)\pi(3)^2$ = $45\pi w$, where w is the density of water.

But what about the force on **the sides** of the can? The pressure is different at different depths. The forces on the sides will be different at all the many different values of h between h=0 and h=5. We need an integral to add them all up.

Choose a typical y between y=0 and y=5 (below). Then the depth at that y is (5-y) and the pressure there is $\rho(5-y)$. The fluid at this depth exerts a force on a (circular area) of width dy. Call the area of this circular area dA, for differential area.

$$dA = 2\pi(\text{radius})dy$$
$$= 2\pi(3)dy$$

The force on the sides at this depth is (pressure)(area) = $\rho(5-y)dA$. Adding up all the forces from bottom to top, we have:

$$\text{total force} = \int_0^5 \rho(5-y)dA = \int_0^5 \rho(5-y)2\pi(3)dy$$
$$= 6\pi\rho\int_0^5(5-y)dy$$

The general formula is
$$\boxed{F = \int(\text{density})(\text{depth})dA}$$

PROBLEM: Find the total force exerted by a lake 18 ft. deep
on a dam of the dimensions shown.

Consider a typical y between 0 and 18 (below):

Note that it doesn't matter how long the lake is, only how
deep it is at the dam.

The depth at y is 18-y. The pressure = (density)(depth)
= w(18-y).dA = 2xdy. Since $y = \frac{18x}{5}$, $x = \frac{5y}{18}$. So, $dA = 2(\frac{5y}{18})dy$.

$$F = \int_0^{18} (\text{density})(\text{depth})dA$$

$$= \int_0^{18} w(18-y)(2)(\frac{5y}{18})dy$$

By the way, $w = 62.5$ $lb/_{ft}3$.

VI. WORK

Work = (force)(distance).

So, if a brick falls 17 ft., then gravity does 170
ft-lbs of work. If you pull a brick 17 ft. straight up with
a rope, then you do 170 ft-lbs of work against the force of
gravity.

The work is easy to calculate in these problems because
the force of gravity can be assumed to be constant over short
distances. What happens if the force varies with position?

For instance, suppose a 10 lb. bag of sand is raised 17
feet, but sand is leaking out of it at the constant rate of
$\frac{1}{8}$ lb. per foot. When the bag is located at any y between 0 and
17 (see below), it weighs $10 - \frac{y}{8}$ lbs. A force of $10 - \frac{y}{8}$ lbs.
is required to raise the bag the short distance dy:

The work required to raise the bag dy = (force)(distance)
= $(10 - \frac{y}{8})dy$.

The total work to raise the bag through every dy between
y = 0 and y = 17 is

$$\text{Work} = \int_0^{17} (\text{force})(\text{distance}) = \int_0^{17} (10-\frac{y}{8})dy$$

PROBLEM: Revolve $y=x^2$ around the y-axis to get a bowl. Fill it with water and calculate the work required to pump the water to a point 6 feet above the top of the bowl.

Look at a disc of water at a typical y between 0 and 4:

This disc must be raised 10-y feet.

Work to raise disc = (distance)(weight)

[weight = (density) x (volume)]

$= (10-y)(w\pi(\text{radius})^2 dy)$

$= (10-y)(w\pi x^2 dy) = (10-y)(w\pi y dy)$

Work to raise all the discs between y=0 and y=4:

$$W = \int_0^4 (10-y)w\pi y\, dy$$

VII. CENTER OF MASS PROBLEMS

Consider a mass m located a distance x from an axis y:

Then its __moment__ around the y-axis is defined to be

(mass)(distance) = mx.

If there are several masses m_i (i = 1,....,n) located at several distances x_i (i = 1,.....,n), then the __center of mass__ of the system is defined to be:

$$\bar{x} = \frac{\text{sum of moments}}{\text{total weight}} = \frac{\sum_i^n m_i x_i}{\sum_i^n m_i}$$

There is a similar number $\bar{y}$ defined for distances y_i to the x-axis. Simply replace the x_i's by y_i's above.

So, $(\bar{x}, \bar{y})$ would be the x and y coordinates of the center of mass in two dimensions.

In three dimensions, the point $(\bar{x}, \bar{y}, \bar{z})$ is defined analagously, using the coordinates (x_i, y_i, z_i) of the location of each mass m_i.

You can now find the center of mass of any finite collection of point masses. But how can we find it for a continous object like a wire or a plate which contains an infinite number of points? Answer: divide the object up into an infinite number of dm's (differential masses) and use an $\int$ instead of a Σ in the above formulas.

EXAMPLE:

Find the center of mass of the plate of uniform density shown below:

First, divide the plate into small pieces. A typical one is shown below: (called dm for differential mass)

[The words uniform and centroid imply $\rho = 1$.]

weight of small piece $= dm = $ (density)(area)

$= (1)(ydx) = x^2 dx$

In order to calculate the moment for this dm, we need to know how far it is from the x- and y-axes. We will assume that the mass of the dm is concentrated at its center: $(\tilde{x}, \tilde{y})$.

(This is the last piece of notation you have to learn for center of mass, so hang in there.)

$(\tilde{x}, \tilde{y})$ is the midpoint of the dm:

In this case, $\tilde{x} = x$ and $\tilde{y} = \frac{y}{2}$ (half way up).

Since the dm is viewed as concentrated at $(\tilde{x}, \tilde{y})$, then $\tilde{x}$ is the distance from dm to the y-axis and $\tilde{y}$ is the distance to the x-axis:

So, the moment about y is (distance)(mass) $= \tilde{x}dm$ and the moment about x $= \tilde{y}dm$, and the formulas for center of mass are:

$$\bar{x} = \frac{\int \tilde{x}dm}{\int dm} \qquad \bar{y} = \frac{\int \tilde{y}dm}{\int dm} \qquad \bar{z} = \frac{\int \tilde{z}dm}{\int dm}$$

These formulas work for all center of mass problems in 1, 2, and 3 dimensions as long as $(\tilde{x}, \tilde{y}, \tilde{z})$ is the center of mass of the typical dm.

We now finish our example:

$$\bar{x} = \frac{\int_0^1 \tilde{x}\,dm}{\int_0^1 dm} = \frac{\int_0^1 x\,\rho dA}{\int_0^1 \rho dA} \overset{(\rho=1)}{=} \frac{\int_0^1 x\,(x^2)dx}{\int_0^1 x^2 dx} = \frac{3}{4}$$

$$\bar{y} = \frac{\int_0^1 \tilde{y}\,dm}{\int_0^1 dm} = \frac{\int_0^1 \frac{x^2}{2}\rho dA}{\int_0^1 \rho dA} = \frac{\int_0^1 \frac{x^2}{2}(x^2)dx}{\int_0^1 x^2 dx} = \frac{3}{10}$$

Let's do the same problem with different dm's. (You can use any dm's which are uniform in density, so that the $\tilde{x}, \tilde{y}, (\tilde{z})$ will be in the center.)

$(\tilde{x},\tilde{y}) = \left(\frac{x+1}{2}, y\right)$

[midpoint formula]

$$\bar{x} = \frac{\int_0^1 \tilde{x}\,dm}{\int_0^1 dm} = \frac{\int_0^1 \frac{x+1}{2}(1-x)dy}{\int_0^1 (1-x)dy} = \frac{\int_0^1 \frac{1+\sqrt{y}}{2}(1-\sqrt{y})dy}{\int_0^1 (1-\sqrt{y})dy} = \frac{3}{4}$$

(You can do $\bar{y}$.)

PROBLEM: Suppose you had a wire of uniform density:

Find its center of mass.

Since mass = (density)(length), dm = ρds in the case of a wire. And $(\tilde{x},\tilde{y}) = (x,y)$:

$(\tilde{x},\tilde{y}) = (x,y)$

$$\bar{x} = \frac{\int_1^3 x\sqrt{1+4x^2}\,dx}{\int_1^3 \sqrt{1+4x^2}\,dx} \qquad \bar{y} = \frac{\int_1^3 y\sqrt{1+4x^2}\,dx}{\int_1^3 \sqrt{1+4x^2}\,dx} = \frac{\int_1^3 x^2\sqrt{1+4x^2}\,dx}{\int_1^3 \sqrt{1+4x^2}\,dx}$$

$(\bar{x},\bar{y})$ is not on the wire, in this case.

[Note: In 3-D problems, there will of course be a $\bar{z}$ and the dm's will themselves be 3-D. For instance, in order to find the center of mass of a cone, the dm's would be frustrums and dm = ρdV.]

[Sometimes symmetry can be used. A cone is symmetric about the z-axis, so $\bar{x}$ and $\bar{y}$ are zero.]

EXAM PROBLEM: Find the center of mass of a plate bounded
by the curves $x-y^2+3=0$ and $x-2y=0$ if its density is
proportional to distance from the x-axis.

($\rho = ky$. The density is not 1 in this problem.)

We drew this plate earlier in the module:

We must choose <u>horizontal</u> dm's so that y, and therefore

the density, will remain uniform. Then $(\tilde{x}, \tilde{y})$ will be the

midpoint:

Plug
in

$$\tilde{x} = \text{midpoint} = \frac{\text{right} + \text{left}}{2} = \frac{\text{line} + \text{curve}}{2} = \frac{2y + y^2 - 3}{2}$$

$$\tilde{y} = y$$

$$dm = \rho dA = (ky)(2y - (y^2-3)dy)$$

$$\bar{x} = \frac{\int_{-1}^{3} \tilde{x}\, dm}{\int_{-1}^{3} dm} \qquad \bar{y} = \frac{\int_{-1}^{3} \tilde{y}\, dm}{\int_{-1}^{3} dm} \qquad \begin{array}{l}\text{[Note: the} \\ \text{constant k} \\ \text{will cancel.]}\end{array}$$

LSC Mathematics Learning Module VIII

EXERCISES (WITH SOLUTIONS)

compiled by Mathematics Support Capsules, 8/81

I. AREAS

1. Find the area bounded by the parabola $y = 6-x-x^2$ and the x-axis.

2. Find the area bounded by the x-axis, the graph $y = x^2-2x$ and the lines $x = -1$ and $x = 4$.

3. Find the area between the curves
$$y = -x^2-2x+1$$
$$y = (x-1)^2$$

II. VOLUMES

4. A solid has as its base the region between the parabolas $x = y^2$ and $x = -2y^2+3$. Find its volume when rotated about the x-axis.

5. Find the volume of the solid generated by rotating the graph of $y = \frac{1}{x}$ around the x-axis between the points $x = 1$ and $x = 5$.

6. The circle, $x^2+y^2 = a^2$ is rotated about the x-axis to form a sphere. Then a hole of diameter a is bored through the center of the sphere. Find the remaining volume.

III. ARC LENGTH

7. Find the length of the curve
$$y = \frac{1}{3}(x^2+2)^{3/2} \quad \text{for} \quad -2 \le x \le 3$$

8. Set up the integral for the length of the portion of the circle $x^2+y^2 = 4$ for $-1 \le y \le 1$.

IV. SURFACE AREA

9. Find the surface area swept out by rotating the curve $y = \cos x$ for $0 \le y \le 2\pi$ about the line $y = \frac{-6}{5}$. (Set up the integral only.)

10. Find the surface area of the shape formed by rotating the graph of $y = \sqrt{x}$ around the x-axis for $1 \le x \le 6$.

For more problems, see Mathematics Support Center Handout on <u>Bridging the Gap:</u> 111-191, or go directly to Thomas and Finney, <u>Calculus and Analytic Geometry</u>.

SOLUTIONS:

1. <u>DRAW</u>

to find x intercepts,
factor:
  $y = 6-x-x^2$
  $y = (3+x)(2-x)$
  $y = 0$ when $x = -3,2$

So, using vertical lines, we have $\int_{-3}^{2} y \, dx$

$$\int_{-3}^{2} (6-x-x^2)dx = 6x - \frac{x^2}{2} - \frac{x^3}{3} \Big]_{-3}^{2} = 20 \ 5/6$$

2. DRAW

$y = x^2 - 2x$

$($ parabola —

to draw it quickly, put it in standard form by completing the square

$y = (x^2 - 2x + 1) - 1$

$y = (x-1)^2 - 1$

x-intercepts: $0 = x^2 - 2x$, $x^2 = 2x$

$\boxed{x = 2 \quad x = 0}$

The area appears in three separate parts:

Region A: $\int_{-1}^{0} y\, dx = \int_{-1}^{0} (x^2 - 2x)\, dx = \frac{x^3}{3} - x^2 \Big|_{-1}^{0} = \frac{4}{3}$

Region B: $\int_{0}^{2} y\, dx$

$\int_{0}^{2} 0 - (x^2 - 2x)\, dx = -(-\frac{4}{3}) = \frac{4}{3}$

Region C: $\int_{2}^{4} y\, dx = \int_{2}^{4} x^2 - 2x = \frac{20}{3}$

Total area $= \frac{4}{3} + \frac{4}{3} + \frac{20}{3} = \frac{28}{3}$

3. DRAW

to find the limits of integration for x, find where the graphs intersect.

$-x^2 + 2x + 1 = (x-1)^2$

$-x^2 + 2x + 1 = x^2 - 2x + 1$

$4x = 2x^2 \quad \boxed{x = 2, x = 0}$

using vertical slices, evaluate

$\int_{0}^{2} [(\text{top curve}) - (\text{bottom curve})]\, dx$

$\int_{0}^{2} [(-x^2 + 2x + 1) - (x^2 - 2x + 1)\, dx = \int_{0}^{2} (-2x^2 + 4x)\, dx$

$= \frac{-2x^3}{3} + 2x^2 \Big]_{0}^{2} = 8/3$

4. DRAW

to find pt. of intersection:

$y^2 = -2y^2 + 3$

$3y^2 = 3$

$y = \pm 1$

plug in & solve for x

$x = 1^2 \quad x = 1$

$x = -2(1)^2 + 3$

$x = 1$

intersection occurs at (1,1) and (1,-1)

Solve in 2 parts using discs

A: $\int_0^1 \pi y^2 dx$ since $x = y^2$, $\int_0^1 \pi x \, dx = \frac{\pi}{2}$

B: $\int_1^3 \pi y^2 dx$ since $x = -2y^2 + 3$ $\int_1^3 \pi(\frac{x-3}{2}) dx$

$y^2 = -\frac{x-3}{2}$

$= \frac{\pi}{2}\int_1^3 (x-3) dx = -\frac{\pi}{2}[\frac{x^2}{2} - 3x]_1^3 = \pi$

Volume $= A + B = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$

5. DRAW

using disks

$\int_1^5 y^2 \pi \, dx = \pi \int_1^5 \frac{1}{x^2} dx = \pi[-\frac{1}{x}]_1^5 = \frac{4}{5}\pi$

6. Try and visualize what is happening.

Draw a picture.

$x^2 + y^2 = a^2$ is a circle with

radius = a

diameter = 2a

overhead view $\longrightarrow$

Try cylindrical shell method, rotating around the y axis, shells have thickness dx.

Using the formula

$V = \int 2\pi(\text{radius})(\text{height}) dx$

the radius = x   Why?

height = 2y $x^2 + y^2 = a^2$

$V = \int 2\pi \, 2x \, y \, dx$ $y = \sqrt{a^2 - x^2}$

for limits of integration, the shell we are evaluating has an outer radius of a and an inner radius of a/2 so x varies from a to a/2

$V = \int_{a/2}^a 4\pi x \sqrt{a^2 - x^2} \, dx$

To evaluate, let $u = a^2 - x^2 \quad du = -2x \, dx$

$4\pi \int_{x=\frac{a}{2}}^{x=a} -\frac{1}{2} u^{1/2} du = -\frac{4}{3}\pi \, u^{3/2}\Big|_{x=\frac{a}{2}}^{x=a} = -\frac{4}{3}\pi(a^2-x^2)^{3/2}\Big|_{a/2}^a = \frac{\sqrt{3}}{2}\pi a^3$

7. $\frac{dy}{dx} = x(x^2+2)^{1/2}$ $(\frac{dy}{dx})^2 = x^2(x^2+2)$

   $ds = \int \sqrt{1 + (\frac{dy}{dx})^2} \, dx = \int_{-2}^{3} \sqrt{1+x^4+2x^2} \, dx = \int_{-2}^{3} \sqrt{(x^2+1)^2} \, dx$

   $= \int_{-2}^{3} (x^2+1) \, dx = \frac{x^3}{3} + x \Big|_{-2}^{3} = 12 - (-\frac{8}{3} - 2) = \frac{50}{3}$

8. $x^2+y^2 = 4$ $x^2 = 4-y^2$ $x = \sqrt{4-y^2}$ $\frac{dx}{dy} = -y(4-y^2)^{-1/2}$

   $(\frac{dx}{dy})^2 = \frac{y^2}{4-y^2}$ $\boxed{ds = \int_{1}^{1} \sqrt{1 + \frac{y^2}{4-y^2}} \, dy}$

9. DRAW

rotate the curve
obtain a sideways hourglass

radius = $\cos x - (-\frac{6}{5})$

   $= \cos x + \frac{6}{5}$

$ds = \sqrt{1 + (\frac{dy}{dx})^2} \, dx$

$y = \cos x$ $\frac{dy}{dx} = -\sin x$ $(\frac{dy}{dx})^2 = \sin^2 x$

$ds = \sqrt{1 + \sin^2 x} \, dx$ $SA = \int_{0}^{2\pi} 2\pi(\cos x + \frac{6}{5})(\sqrt{1 + \sin^2 x}) \, dx$

10. DRAW

radius = $y = \sqrt{x}$

$ds = \int \sqrt{1 + (\frac{dy}{dx})^2} \, dx$ $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ $(\frac{dy}{dx})^2 = \frac{1}{4x}$ $ds = \int \sqrt{1 + \frac{1}{4x}} \, dx$

$SA = \int_{1}^{6} 2\pi\sqrt{x}(\sqrt{1 + \frac{1}{4x}}) \, dx = \int_{1}^{6} 2\pi\sqrt{x}(\frac{\sqrt{4x+1}}{4x}) \, dx = \int_{1}^{6} 2\pi\sqrt{x}(\frac{\sqrt{4x+1}}{\sqrt{4x}})$

$= \int_{1}^{6} \pi\sqrt{4x+1} \, dx$

using $u = 4x+1$ $du = 4 \, dx$, obtain

$\int_{5}^{25} \frac{\pi}{4} u^{1/2} du = \frac{\pi}{4}(\frac{2}{3} u^{3/2}) \Big|_{5}^{25} = \frac{\pi}{6}(125 - \sqrt{125})$

Note: bounds in integration change if $x = 6$ $u = 4(6)+1 = 25$ etc.

MATHEMATICS LEARNING MODULE IX
TECHNIQUES OF INTEGRATION

by

Raymond W. Bacon

CONTENTS

This module summarizes, for easy reference, all the basic TECHNIQUES OF INTEGRATION normally required for Math 112 and 192. One or two (fairly hard) examples are given for each type. You should do at least ten of each type from your text as you learn each technique. Then, before the exam, do another thirty to fifty (or more!) chosen at random from the review section at the end of the chapter. You should develop the ability to decide quickly which technique to use on a given problem as well as how to calculate the answer. Remember, you can always check your answer by differentiating it to get the function inside the integral.

STRATEGY: Basically, solving integrals by these techniques is a mechanical process that doesn't require much cleverness or aptitude, but rather more practice and familiarity. So don't blow the easy part of the course. Math 112 and 192 are hard. It's probably going to take everything you've got to get a good grade. Since integrals are a large part of the first exam, think of them as a relatively easy opportunity to get a high grade and develop momentum.

1 DIFFERENTIATION FORMULAS

Make sure you know all the differentiation formulas perfectly. If you have a differentiation formula, then (by the Fundamental Theorem) you have an integral formula. [For instance, if you know that $\frac{d}{dx}(\tan x) = \sec^2 x$, then you know that $\int \sec^2 x\, dx = \tan x + C$.] All of the techniques that follow are used to simplify complex integrals so you can use the basic formulas. You must have the basic formulas at your fingertips in order to "see" the simplified form which you are aiming for. If you don't know which formulas you are required to know, then FIND OUT. This is one of the few times that straight memorization will help you at all in mathematics, so DO IT.

2 U-SUBSTITUTIONS

Know exactly how to do a U-substitution. This is the basic technique , which you always want to try first.

EXAMPLE:

$$\int x\sqrt{x^2+1}\; dx \qquad\qquad \text{set}\quad u = x^2+1$$

$$\qquad\qquad\qquad\qquad \frac{du}{dx} = 2x \;\text{(take derivative)}$$

$$= \int x\, u^{\frac{1}{2}}(\frac{du}{2x}) \qquad\qquad dx = \frac{du}{2x}\;\text{(solve for\ \ dx)}$$

(cancel x's)
(bring out constant)

$$= \frac{1}{2}\int u^{\frac{1}{2}}\, du \qquad\qquad\qquad \text{(This is one of the formulas)}$$

$$= \frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C \qquad\qquad\qquad \text{(Plug in\ \ u\ \ to get back to\ \ x's)}$$

$$= \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$$

CHECK!
$$\frac{d}{dx}[\frac{1}{3}(x^2+1)^{\frac{3}{2}}+C]$$
$$= (\frac{1}{3})(\frac{3}{2})(x^2+1)^{\frac{1}{2}}(2x)$$
$$= x\sqrt{x^2+1}$$

Once you've set u equal to the right thing, all you have to do is differentiate, plug in, and use the formulas. It's easy and leads to few mistakes.

But how do you know what to let u be? A good guideline is to let u be something so that u's derivative is also in the integral (ignoring constants) and will cancel out (like the x's did above.)

EXAMPLES:

$\int \sin^2 x \cos x \, dx$ 　　　　set $u = \sin x$

　　　　　　　　　　　　　　　since $\frac{du}{dx} = \cos x$

$\int \frac{t^2}{\sqrt{t^3+1}} \, dt$ 　　　　set $u = t^3+1$

　　　　　　　　　　　　　　　since $\frac{du}{dt} = 3t^2$

EXAMPLE:

$\int \frac{e^{(\sin^{-1}5x)}}{\sqrt{1-25x^2}} \, dx$ 　　You know $\frac{d}{dx}(\sin^{-1}5x) = \frac{1}{\sqrt{1-25x^2}}(5)$

$= \int \frac{e^u \sqrt{1-25x^2} \, du}{\sqrt{1-25x^2}(5)}$ 　　so set $u = \sin^{-1}(5x)$

　　　　　　　　　　　　then $du = \frac{(5)dx}{\sqrt{1-25x^2}}$

$= \frac{1}{5}\int e^u du$

　　　　　　　　　　　　　　　$dx = \frac{\sqrt{1-25x^2} \, du}{5}$

$= \frac{e^u}{5} + C$

$= \frac{e^{\sin^{-1}5x}}{5} + C$

(Notice, you don't get cancellation for $\int \sqrt{1-25x^2} \, e^{(\sin^{-1}5x)}dx$ with this u-substitution. You end up with $\int (1-25x^2)e^u du$. You have to get rid of <u>all</u> the x's or the u-substitution won't work.

This integral is much harder. You might try to come back to it after you've finished the module.)

Sometimes, a u-substitution will work even if the cancellation isn't quite complete:

$\int \frac{t^5}{\sqrt{t^3+1}} \, dt$ 　　　　$u = t^3+1$

　　　　　　　　　　　　　$\frac{du}{dt} = 3t^2$

$= \int u^{-\frac{1}{2}} t^5 \, \frac{du}{3t^2}$ 　　$dt = \frac{du}{3t^2}$

$= \frac{1}{3}\int u^{-\frac{1}{2}} t^3 du$

We haven't gotten rid of all the t's yet, but I can look back at my original substitution and solve for t^3 in terms of u: $t^3 = u-1$.

Now

$= \frac{1}{3}\int u^{-\frac{1}{2}}(u-1)du$

$= \frac{1}{3}(\int u^{+\frac{1}{2}}du - \int u^{-\frac{1}{2}}du)$

$= \frac{1}{3}(\frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{1}{2}}}{\frac{1}{2}}) + C$

$= \frac{2}{9}(t^3+1)^{\frac{3}{2}} - \frac{2}{3}(t^3+1)^{\frac{1}{2}} + C$

This type of integral is dealt with in more depth later in the module under <u>rationalizing</u> <u>substitutions</u>.

For now, make sure you have the ordinary u-substitution well under control before going on. All calculus texts have lots of u-substitution problems.

3 Remember $\boxed{\sin^2 x + \cos^2 x = 1}$

Divide by $\sin^2 x \rightarrow \boxed{1 + \cot^2 x = \csc^2 x}$

or

Divide by $\cos^2 x \rightarrow \boxed{\tan^2 x + 1 = \sec^2 x}$

Use these on a) <u>odd</u> powers of sin & cos

b) <u>any</u> powers of tan & cot (power $\geq$ 2)

c) <u>even</u> powers of sec & csc

EXAMPLE:

$\int \csc^4 2x \, dx$

(Why have I done this? Because I can change csc's to cot's and the derivatives of cot's are $\csc^2$'s, which will cancel out in a u-substitution.)

$= \int \csc^2 2x \, \csc^2 2x \, dx$

$= \int (1 + \cot^2 2x) \csc^2 2x \, dx$

$= \int \csc^2 2x \, dx + \int \cot^2 2x \, \csc^2 2x \, dx$

$= -\dfrac{\cot 2x}{2} - \dfrac{\cot^3 2x}{3(2)} + C$

4 Memorize $\sin^2 x = \dfrac{1}{2} - \dfrac{1}{2} \cos 2x$

$\cos^2 x = \dfrac{1}{2} + \dfrac{1}{2} \cos 2x$

Use these on <u>even</u> powers of sin + cos

EXAMPLE:

$\int \sin^4 \dfrac{t}{2} \, dt$

$= \int (\sin^2 \dfrac{t}{2})^2 dt$

$= \int (\dfrac{1}{2} - \dfrac{1}{2} \cos t)^2 dt$

$= \int \dfrac{1}{4} dt - \dfrac{\int \cos t \, dt}{2} + \int \dfrac{1}{4} \cos^2 t \, dt$

$= \dfrac{t}{4} - \dfrac{\sin t}{2} + \dfrac{1}{4}(\int (\dfrac{1}{2} + \dfrac{1}{2} \cos 2t) dt)$

$= \dfrac{t}{4} - \dfrac{\sin t}{2} + \dfrac{t}{8} + \dfrac{\sin 2t}{16} + C$

5 Memorize the three "trig-substitutions" and be able to draw the Δ's that go with them

$\boxed{\begin{array}{l} \text{for} \\ a^2 - u^2 \rightarrow u = a \sin \theta \\ \text{for} \\ a^2 + u^2 \rightarrow u = a \tan \theta \\ \text{for} \\ u^2 - a^2 \rightarrow u = a \sec \theta \end{array}}$

$\sin \theta = \dfrac{u}{a}$

$\tan \theta = \dfrac{u}{a}$

$\sec \theta = \dfrac{u}{a}$

EXAMPLE:

$\int \dfrac{dx}{\sqrt{9 + 4x^2}}$

$a = 3$ $\qquad$ $u = a \tan \theta$

$u = 2x$ $\qquad$ $2x = 3 \tan \theta$

$dx = \dfrac{3}{2} \sec^2 \theta \, d\theta$

I need $\sqrt{9 + 4x^2}$

$\sqrt{9 + 4x^2} = \sqrt{9 + (3 \tan \theta)^2}$

(since $2x = 3 \tan \theta$)

$= \sqrt{9(1 + \tan^2 \theta)}$

$= \sqrt{9 \sec^2 \theta}$

$= 3 \sec \theta$

$= \int \dfrac{\frac{3}{2} \sec^2 \theta \, d\theta}{3 \sec \theta}$

$= \dfrac{1}{2} \int \sec \theta \, d\theta$

($\int \sec x \, dx = \ln|\sec x + \tan x|$ is an extra formula which you will want to memorize)

$= \dfrac{1}{2} \ln|\sec \theta + \tan \theta| + C$

$$= \frac{1}{2} \ln\left|\frac{\sqrt{9+4x^2}}{3} + \frac{2x}{3}\right| + c$$

$$2x = 3\tan\theta$$

$$\tan\theta = \frac{2x}{3}$$

(Note: If you had gotten a θ in the answer then solve for θ in $\tan\theta = \frac{2x}{3}$ to get $\theta = \tan^{-1}(\frac{2x}{3})$.)

$\boxed{6}$ Complete the Square

Use it when you've got an ax^2+bx+c (often in a radical in the denominator) you can't get rid of. The idea is to change ax^2+bx+c to $a^2 \pm u^2$ or u^2-a^2 and then use trig substitutions.

[Note: The two a's are not the same here.]

METHOD:

$$ax^2+bx+c = (x+\frac{b}{2a})^2 + (\text{whatever you need to make } c \text{ come out right})$$

$\left[\text{then } x + \frac{b}{2a} \text{ will be } u.\right]$

EXAMPLE:

$$\int \frac{dx}{\sqrt{x^2-2x-8}}$$

$$= \int \frac{dx}{\sqrt{(x-1)^2-9}}$$

$$= \int \frac{3\sec\theta\tan\theta\, d\theta}{3\tan\theta}$$

$$\frac{b}{2a} = -1$$

Now we have u^2-a^2 with

$u = x-1,\ a = 3$

$u = a\sec\theta$

$x-1 = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\, d\theta$

$\sqrt{(x-1)^2-9} = 3\tan\theta$

$$= \ln|\sec\theta+\tan\theta| + c$$

$$= \ln\left|\frac{x-1}{3} + \frac{\sqrt{(x-1)^2-9}}{3}\right| + c$$

A common sense attack:

$$\int \frac{x-1}{\sqrt{x^2-4x+3}}\, dx$$

$$= \frac{1}{2}\int \frac{2x-2}{\sqrt{x^2-4x+3}}\, dx$$

$$= \frac{1}{2}\int \frac{2x-4+2}{\sqrt{x^2-4x+3}}\, dx$$

$$= \frac{1}{2}\int \frac{2x-4}{\sqrt{x^2-4x+3}}\, dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2-4x+3}}$$

$$= (\text{an ordinary u-substitution}) + (\text{complete the square})$$

(If you were going to use a u-substitution, then you would need a $2x-4$ in the top, so multiply and divide by 2 and subtract and add 2)

[You could have simply completed the square from the beginning:]

$$\int \frac{x-1}{\sqrt{x^2-4x+3}}\, dx$$

$$= \int \frac{x-1}{\sqrt{(x-2)^2-1}}\, dx$$

so

$u = x-2$

$a = 1$

$x-2 = \sec\theta$

$dx = \sec\theta\tan\theta\, d\theta$

and

$x-1 = \sec\theta+1$

[7] PARTIAL FRACTIONS

Partial fractions are hard to explain, but fairly easy to catch on to simply by watching a few solutions. Use this technique on <u>quotients</u> of <u>polynomials</u>. <u>First, make sure the degree of the numerator is less than the degree of the denominator</u>. (If not, divide first.)

The idea is to factor the denominator and then divide the fraction up into a sum of fractions that are easier to integrate, for instance:

$$\int \frac{1}{x^2-1}\, dx$$

$$= \int \frac{1}{(x-1)(x+1)}\, dx$$

$$= \int \frac{\frac{1}{2}}{(x-1)}\, dx - \int \frac{\frac{1}{2}}{(x-1)}\, dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

But how do you divide the fraction in the integral into a sum of simpler fractions?

METHOD:

Factor the denominator into powers of linear and quadratic terms. (There is a theorem that says you can do this.)

[NOTE: In what follows A, B, C, D, etc. are constants

AX+B, CX+D, EX+F, etc. are linear terms

AX^2+B, CX^2+DX+E, etc. are quadratic terms

(   ) is a factor which contains some stuff that doesn't matter for the moment]

So, for instance, you might have

$$\int \frac{(\quad)}{(x+1)(2x-3)(x^2+1)(7x^2+3x+1)(x-2)^2(x^2+7)^3}\, dx$$

(Most real problems are nowhere near this complex)

1) Each <u>linear</u> term gets a $\frac{(constant)}{(linear)}$.

So $$\frac{(\quad)}{(x+1)(2x-3)(x^2+1)(7X^2+3x+1)(x-2)^2(x^2+7)^3}$$

$$= \frac{A}{x+1} + \frac{B}{2x-3} + (\text{other stuff}).$$

2) Each <u>quadratic</u> term gets a $\frac{(linear)}{(quadratic)}$

$$= \frac{A}{x+1} + \frac{B}{2x-3} + \frac{CX+D}{x^2+1} + \frac{EX+F}{7x^2+3x+1} + (\text{other stuff})$$

3) If the <u>linear</u> or <u>quadratic</u> is raised to a power p. Then you do either 1 or 2 , above, <u>p times</u> as follows:

$$\frac{(\quad)}{(x-2)^2} = \frac{G}{x-2} + \frac{H}{(x-2)^2}$$

⎡2 fractions, increasing exponent in denominator each time⎤

$$\frac{(\quad)}{(x^2+7)^3} = \frac{IX+J}{x^2+7} + \frac{KX+L}{(x^2+7)^2} + \frac{MX+N}{(x^2+7)^3}$$

⎡3 fractions increasing exponent in denominator each time.⎤

Now our original integral

$$\int \frac{(\quad)}{(x+1)(2x-3)(x^2+1)(7x^2+3x+1)(x-2)^2(x^2+7)^3}$$

$$= \int \frac{A}{x+1}\,dx + \int \frac{B}{2x-3}\,dx + \int \frac{CX+D}{x^2+1}\,dx + \frac{EX+F}{7x^2+3x+1}\,dx$$

$$+ \int \frac{G}{x-2}\,dx + \int \frac{H}{(x-2)^2}\,dx + \int \frac{IX+J}{x^2+7}\,dx + \int \frac{KX+L}{(x^2+7)^2} + \int \frac{MX+N}{(x^2+7)^3}$$

You can now do each of these simpler integrals if you know the constants A, B, C, D, E,... etc. We will show how to evaluate the constants in the course of doing the next example. (As noted before, it's a lot easier to do partial fractions problems than it is to talk about doing them.)

EXAMPLE:

$$\int \frac{(x-1)}{(x+1)(x^2+1)}\,dx = \int \left(\frac{A}{x+1}\right)dx + \int \left(\frac{BX+C}{x^2+1}\right)dx$$

To evaluate A, B, and C, you now combine the fractions.

$$\frac{(x-1)}{(x+1)(x^2+1)} = \frac{A(x^2+1)+(BX+C)(x+1)}{(x+1)(x^2+1)} \;.$$

The numerators are equal.

$$x-1 = A(x^2+1)+(BX+C)(x+1)$$
$$= Ax^2+A+BX^2+BX+CX+C$$
$$(0)x^2+(1)x-1 = (A+B)x^2+(B+C)x+(A+C) \quad \text{[Set coefficients equal.]}$$

So A+B = 0

B+C = 1

A+C = -1.

Solve simultaneously to get A = -1

B = 1

C = 0.

Now

$$\int \frac{(x-1)}{(x+1)(x^2+1)}\,dx$$

$$= \int \frac{-1}{x+1}\,dx + \int \frac{x}{x^2+1}\,dx$$

$$= -\ln|x+1| + \frac{\ln|x^2+1|}{2} + C$$

SHORT CUT for evaluating A, B, and C:

The equation $x-1 = A(x^2+1)+(BX+C)(x+1)$ is true for all x's, so substitute in some specific x's that will make the expression simple.

If x = -1 then

$$(-1)-1 = A(1+1) + 0$$
$$-2 = 2A$$
$$A = -1.$$

If x = 0

$$0-1 = A(1) + (B(0)+C)(0+1)$$
$$-1 = A + C$$
$$-1 = -1 + C \quad \text{(since A = -1)}$$
so C = 0.

Now you have

$$x-1 = -1(x^2+1) + (BX)(x+1).$$

Choose, say, x = 1

$$0 = -1(1+1) + B(1+1)$$
$$0 = -2 + 2B$$
$$2B = 2$$
$$B = 1.$$

Here are two more examples:

$$\int \frac{2x^2+3}{x^2(x-1)} \, dx$$

$$= \int \frac{AX+B}{x^2} \, dx + \int \frac{C}{x-1} \, dx \qquad\qquad A = -3$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad B = -3$$
$$= -3 \, \ell n|x| + \frac{3}{x} + 5 \, \ell n|x-1| + C$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad C = 5$$

$$\int \frac{dx}{(x^3-1)}$$

$$= \int \frac{dx}{(x-1)(x^2+x+1)}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad A = \frac{1}{3}$$
$$= \int \frac{A}{x-1} \, dx + \int \frac{BX+C}{x^2+x+1} \, dx$$
$$\qquad\qquad\qquad\qquad\qquad\qquad B = -\frac{1}{3}$$
$$= \frac{1}{3} \, \ell n|x-1| + \text{(do a u-substitution} \qquad C = -\frac{2}{3}$$
$$\qquad\qquad\qquad \text{and/or complete the}$$
$$\qquad\qquad\qquad \text{square)}$$

As you can see, partial fractions integrals make good test questions because you often have to use the other techniques you have learned in order to work out the new simpler integrals.

[8] By Parts

Memorize $\int u\,dv = uv - \int v\,du$. You have to decide what u is going to be, then the rest of the stuff in the integral is automatically dv. The best way to learn what to let u be is by practicing. You decide whether you've made a good choice for u by looking at whether $\int v\,du$ on the right is simpler than $\int u\,dv$ on the left. Look at the homework problems in the "by parts" section of your book to get an idea of what kinds of integrals need this method.

$$\int x \sin x \, dx \qquad\qquad \begin{bmatrix} u = x & du = dx \\ v = -\cos x & dv = \sin x \, dx \end{bmatrix}$$

$$\int x \sin x \, dx$$

$$= \int u \, dv$$

$$= uv - \int v \, du$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x$$

CHECK

$$\frac{d}{dx}(-x\cos x + \sin x) = (-x)(-\sin x) + (\cos x)(-1) + \cos x = x \sin x$$

Sometimes you have to use "by parts" twice.

$$\int e^x \sin x \, dx \qquad\qquad u = e^x \qquad du = e^x dx$$
$$\qquad\qquad\qquad\qquad\qquad v = -\cos x \quad dv = \sin x \, dx$$
$$= \int u \, dv = uv - \int v \, du$$

$$= -e^x\cos x - \int(-\cos x)e^x dx$$

$$= -e^x\cos x + \int e^x \cos x \, dx \qquad \text{Do "by parts" again on this integral}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad u_1 = e^x \qquad du_1 = e^x dx$$
$$(u_1 v_1 - \int v_1 du_1) \qquad v_1 = \sin x \quad dv_1 = \cos x \, dx$$

$$= -e^x\cos x + (e^x\sin x - \int \sin x \, e^x dx)$$

We can now solve for our original integral in

$$\int e^x \sin x \, dx = -e^x\cos x + e^x\sin x - \int e^x\sin x \, dx$$

$$2\int e^x \sin x \, dx = -e^x \cos \cdot x + e^x \sin x$$

$$\int e^x \sin x \, dx = \frac{-e^x \cos x + e^x \sin x}{2} + C$$

Check: $\dfrac{d}{dx}(\dfrac{-e^x \cos x + e^x \sin x}{2} + C)$

$$= \frac{1}{2}(-e^x(-\sin x) + \cos x(-e^x) + e^x \cos x + \sin x \, e^x) = \frac{1}{2}(2e^x \sin x)$$

$$= e^x \sin x$$

[9] Rational functions of $\sin x$ and $\cos x$ (sometimes optional).

Memorize $z = \tan \dfrac{x}{2}$

$$\sin x = \frac{2z}{1+z^2}$$

$$\cos x = \frac{1-z^2}{1+z^2}$$

$$dx = \frac{2 \, dz}{1+z^2} \ .$$

The method consists of plugging the above into the integral and then working out the horrible arithmetic that usually results. Remember to change z's back to x's at the end.

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{2 + \cos x} \, dx$$

$$= \int \frac{\frac{2z}{1+z^2}}{2 + \frac{1-z^2}{1+z^2}} \left(\frac{2 \, dz}{1+z^2}\right)$$

$$= \int \frac{\frac{2z}{(1+z^2)}}{\frac{2+2z^2+1-z^2}{(1+z^2)}} \left(\frac{2 \, dz}{1+z^2}\right)$$

$$= 4\int \frac{z \, dz}{(3+z^2)(1+z^2)} \qquad \text{(partial fractions)}$$

$$= 4\left(\int \frac{-\frac{1}{2}z}{3+z^2} \, dz + \int \frac{\frac{1}{2}z}{1+z^2} \, dz\right)$$

$$= 4\left(-\frac{1}{4} \ln|3+z^2| + \frac{1}{4} \ln|1+z^2|\right)$$

$$= \ln\left|\frac{1+z^2}{3+z^2}\right|$$

$$= \ln\left|\frac{1+\tan^2 \frac{x}{2}}{3+\tan^2 \frac{x}{2}}\right|_0^{\frac{\pi}{2}}$$

$$= \ln\left|\frac{1+1}{3+1}\right| - \ln\left|\frac{1}{3}\right|$$

$$= \ln\left|\frac{3}{2}\right|$$

[10] Rationalizing Substitutions

Plug in something that will get rid of the trouble. "Rationalizing" means "get rid of the radicals" (or, at least, get them out of the denominator). So, in general, let u be the worst looking radical in the integral and see what happens.

$$\int \frac{dx}{a + b\sqrt{x}} \qquad\qquad u = a + b\sqrt{x}$$

$$du = \frac{b}{2\sqrt{x}} \, dx$$

$$= \frac{2}{b}\int \frac{\sqrt{x} \, du}{u} \qquad\qquad \frac{2\sqrt{x} \, du}{b} = dx$$

You want all u's, no x's.

$$= \frac{2}{b^2}\int \frac{u-a}{u} \, du \qquad\qquad \sqrt{x} = \frac{u-a}{b}$$

$$= \frac{2}{b^2}\left(\int 1 \, du - \int \frac{a}{u} \, du\right)$$

$$= \frac{2}{b^2} (u - a\ln|u|)$$

$$= \frac{2}{b^2} (a+b\sqrt{x} - a\ln|a+b\sqrt{x}|) + C$$

$$2\int \frac{\sqrt{x}\,du}{a+bu}$$

You could also let $u = \sqrt{x}$

$$du = \frac{dx}{2\sqrt{x}}$$

$$=2\int \frac{u\,du}{a+bu}$$

$$dx = 2\sqrt{x}\,du$$

$$= 2(\int \frac{1}{b}du - \frac{a}{b}\int \frac{1}{a+bu}\,du$$

(divide u by $a+bu$ to get $\frac{1}{b} - \frac{a}{\dfrac{b}{a+bu}}$).

$$= \frac{2u}{b} - \frac{2a}{b} \frac{\ln|a+bu|}{b}$$

$$= \frac{2\sqrt{x}}{b} - \frac{2a}{b^2} \ln|a+b\sqrt{x}| \longrightarrow \text{which also checks since it differs}$$
$$+ C \qquad\qquad\qquad \text{from the first answer by a constant}$$

Check

$$\frac{d}{dx}(\frac{2}{b^2}(a+b\sqrt{x} - a\ln|a+b\sqrt{x}|)) = \frac{2}{b^2}(\frac{b}{2\sqrt{x}} - \frac{a}{(a+b\sqrt{x})(2\sqrt{x})})$$

$$= \frac{2}{b^2}(\frac{b(a+b\sqrt{x}) - ab}{(a+b\sqrt{x})(2\sqrt{x})}) = (\frac{2}{b^2})\frac{b^2\sqrt{x}}{(a+b\sqrt{x})\sqrt{x}} = \frac{1}{a+b\sqrt{x}}$$

EXTRA: For some courses you need the following:

$$\sin(mx)\sin(nx) = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

$$\sin(mx)\cos(nx) = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$

$$\cos(mx)\cos(nx) = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

TECHNIQUES OF INTEGRATION

A Quick Review of the different techniques of integration:

1) U-Substitutions

2) the substitutions using
$$\sin^2 x + \cos^2 x = 1$$
$$1 + \cot^2 x = \csc^2 x \quad \Big\} \text{ use for}$$
$$1 + \tan^2 x = \sec^2 x$$

   a) odd powers of sin & cos
   b) any power of tan & cot
   c) even powers of sec & csc.

3) substitution using
$$\sin^2 x = \tfrac{1}{2} - \tfrac{1}{2} \cos 2x \quad \Big\} \text{ use on even powers}$$
$$\cos^2 x = \tfrac{1}{2} + \tfrac{1}{2} \cos 2x \quad \text{ of sin \& cos.}$$

4) Trig Substitutions
   for
   $a^2 - u^2 \Rightarrow$ use $u = a\sin\theta$
   $a^2 + u^2 \Rightarrow$ use $u = a\tan\theta$
   $u^2 - a^2 \Rightarrow$ use $u = a\sec\theta$

   pictures:

   $a^2 - u^2$

   $a^2 + u^2$

   $u^2 - a^2$

5) Completing a square, then using trig substitution

6) Partial fractions

7) By parts

8) Rational functions of sin and cos $\Rightarrow$ sub in $z = \tan\frac{x}{2}$
   the substitutions: $z = \tan\frac{x}{2}$, $\sin x = \frac{2z}{1+z^2}$, $\cos x = \frac{1-z^2}{1+z^2}$, $dx = \frac{2dz}{1+z^2}$

9) Rationalizing Substitutions $\Rightarrow$ getting rid of radicals

Solve:

1) $\int \cos^3 x \, dx$

2) $\int x\sqrt{x^2+1} \, dx$

3) $\int \frac{x+2}{x^2+4x+7} \, dx$

4) $\int \log e^{x^2} \, dx$

5) $\int \frac{1}{x} \log x \, dx$

6) $\int (x^2+1)^3 \, dx$

7) $\int 3^x \, dx$

8) $\int x e^{x^2} \, dx$

9) $\int \cot^4 x \, dx$

10) $\int \tan^3 x \, dx$

11) $\int \sin^4 4x \, dx$

12) $\int \sin^2 x \cot^3 x \, dx$

13) $\int \frac{\sqrt{1-x^2}}{x^4}$

14) $\int \sqrt{1-x^2} \, dx$

15) $\int \frac{dx}{x^2-4}$

16) $\int \frac{2x^2+3}{x(x-1)^2} \, dx$

17) $\int \log x \, dx$

18) $\int \frac{xe^x}{(x+1)^2} \, dx$

19) $\int x\sqrt{x+1} \, dx$

20) $\int \frac{dx}{1+2\cos x}$

21) $\int \frac{\sec x}{2\tan x + \sec x - 1} \, dx$

22) $\int \frac{dx}{1+\sqrt{x}}$

23) $\int \sqrt{1-e^x} \, dx$

24) $\int \frac{dx}{2x^2+8x+20}$

25) $\int \frac{dx}{\sqrt{x^2+x+1}}$

1) $\int \cos^3 x \, dx = \int \cos^2 x \cdot \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx$

$= \int \cos x \, dx - \underbrace{\int \cos x \sin^2 x \, dx}_{u = \sin x, \quad du = \cos x \, dx}$

$= \int \cos x \, dx - \int u^2 \cos x \, \frac{du}{\cos x} = \sin x - \frac{1}{3} u^3 + C$

$= \sin x - \frac{1}{3} \sin^3 x + C$

---

2) $\int x (x^2 + 1)^{1/2} \, dx \qquad u = x^2 + 1 \qquad du = 2x \, dx \qquad \frac{du}{2x} = dx$

$= \int x \, u^{1/2} \, \frac{du}{2x} = \frac{1}{2} \int u^{1/2} \, du = \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} = \frac{1}{3} u^{3/2} + C = \frac{1}{3} (x^2 + 1)^{3/2} + C$

---

3) $\int \frac{x+2}{x^2 + 4x + 7} \, dx \qquad u = x^2 + 4x + 7 \qquad du = (2x + 4) \, dx$

$\int \frac{x+2}{u} \, \frac{du}{2(x+2)} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln|u| = \frac{1}{2} \ln|x^2 + 4x + 7| + C$

---

4) $\int \log e^{x^2} \, dx = \int x^2 \, dx = \frac{x^3}{3} + C$

---

5) $\int \frac{1}{x} \log x \, dx \qquad u = \log x \qquad du = \frac{1}{x} \, dx \qquad x \, du = dx$

$\int \frac{1}{x} u \, x \, du = \int u \, du = \frac{1}{2} u^2 = \frac{1}{2} (\log x)^2 + C$

---

6) $\int (x^2 + 1)^3 \, dx = \int (x^2 + 1)(x^4 + 2x^2 + 1) \, dx = \int (x^6 + 3x^4 + 3x^2 + 1) \, dx$

$= \frac{1}{7} x^7 + \frac{3}{5} x^5 + x^3 + x + C$

---

7) $\int 3^x \, dx \qquad \text{Set } 3^x = e^{\log 3^x} = e^{x \log 3}, \text{ then } \int 3^x \, dx = \int e^{x \log 3} \, dx = \frac{1}{\log 3} e^{x \log 3} + C$

$\frac{1}{\log 3} e^{x \log 3} + C = \frac{1}{\log 3} (3^x) + C$

---

8) $\int x e^{x^2} \, dx \qquad u = x^2 \qquad du = 2x \, dx$

$\int x e^u \, \frac{du}{2x} = \frac{1}{2} \int e^u \, du = \frac{1}{2} e^u = \frac{1}{2} e^{x^2} + C$

---

9) $\int \cot^4 x \, dx = \int \cot^2 x \cot^2 x \, dx = \int \cot^2 x (\csc^2 x - 1) \, dx$

$= \int \cot^2 x \csc^2 x \, dx - \int \cot^2 x \, dx - \int \cot^2 x \csc^2 x \, dx - \int \cot^2 x (\csc^2 x - 1) \, dx$

$= \int \cot^2 x \csc^2 x \, dx - \int \csc^2 x \, dx + \int 1 \, dx$

$u = \cot x \, dx$
$du = -\csc^2 x \, dx$ \qquad follow formulas for these

$= \int u^2 \csc^2 x \, \frac{du}{-\csc^2 x} - \int \csc^2 x \, dx + \int dx$

$= -\frac{1}{3} u^3 + \cot x + x = -\frac{1}{3} \cot^3 x + \cot x + x + C$

---

10) $\int \tan^3 x \, dx$

$= \int \tan x \tan^2 x \, dx = \int \tan x (\sec^2 x - 1) \, dx = \int \tan x \sec^2 x \, dx - \int \tan x \, dx$

$= \int \tan x \sec^2 x \, dx - \int \frac{\sin x}{\cos x} \, dx$

$u = \tan x \qquad\qquad q = \cos x$
$du = \sec^2 x \, dx \qquad dq = -\sin x \, dx$

$= \int u \sec^2 x \, \frac{du}{\sec^2 x} - \int \frac{\sin x}{q} \, \frac{dq}{-\sin x} = \int u \, du + \int \frac{dq}{q}$

$= \frac{1}{2} u^2 + \ln|q| = \frac{1}{2} \tan^2 x + \ln|\cos x| + C$

---

11) $\int \sin^4 4x \, dx \qquad u = 4x \qquad du = 4 \, dx$

$= \frac{1}{4} \int \sin^4 u \, du = \frac{1}{4} \int (\sin^2 u)^2 \, du$

sub in $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2u$

$= \frac{1}{4} \int \left(\frac{1}{2} - \frac{1}{2} \cos 2u \right)^2 du = \frac{1}{4} \int \left(\frac{1}{4} - \frac{1}{2} \cos 2u + \frac{1}{4} \cos^2 2u \right) du$

$= \frac{1}{16} \int du - \frac{1}{8} \int \cos 2u \, du + \frac{1}{16} \int \cos^2 2u \, du$

sub in again $\cos^2 2u = \frac{1}{2} + \frac{1}{2} \cos 4u$

$= \frac{1}{16} \int du - \frac{1}{8} \int \cos 2u \, du + \frac{1}{16} \int \left(\frac{1}{2} + \frac{1}{2} \cos 4u \right) du$

$= \frac{1}{16} \int du - \frac{1}{8} \int \cos 2u \, du + \frac{1}{32} \int du + \frac{1}{32} \int \cos 4u \, du$

$= \frac{1}{16} u - \frac{1}{8} \left(\frac{1}{2} \sin 2u \right) + \frac{1}{32} u + \frac{1}{32} \left(\frac{1}{4} \sin 4u \right)$

$= \frac{3}{32} (4x) - \frac{1}{16} \sin(8x) + \frac{1}{128} \sin(16x) = \frac{3}{8} x - \frac{1}{16} \sin 8x + \frac{1}{128} \sin 16x$

---

12) $\int \sin^2 x \cot^3 x \, dx = \int \sin^2 x \left(\frac{\cos^3 x}{\sin^3 x} \right) dx = \int \frac{\cos^3 x}{\sin x} \, dx = \int \frac{\cos x (1 - \sin^2 x)}{\sin x} \, dx$

$= \int \frac{\cos x}{\sin x} \, dx - \int \cos x \sin x \, dx$

for both, set $u = \sin x, \, du = \cos x \, dx$

$= \int \frac{\cos x}{u} \, \frac{du}{\cos x} - \int \cos x \, u \, \frac{du}{\cos x} = \int \frac{du}{u} - \int u \, du = \ln|u| - \frac{1}{2} u^2$
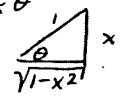
$= \ln|\sin x| - \frac{1}{2} \sin^2 x + C$

---

13) $\int \frac{\sqrt{1 - x^2}}{x^4} \, dx \qquad \text{Trig sub} \qquad a = 1, \quad u = x \qquad x = \sin \theta \qquad dx = \cos \theta \, d\theta$

$= \int \frac{\sqrt{1 - \sin^2 \theta}}{\sin^4 \theta} \cos \theta \, d\theta = \int \frac{\cos \theta}{\sin^4 \theta} \cos \theta \, d\theta = \int \frac{\cos^2 \theta}{\sin^4 \theta} \, d\theta = \int \frac{(1 - \sin^2 \theta)}{\sin^4 \theta} \, d\theta$

$= \int \frac{1}{\sin^4 \theta} \, d\theta - \int \frac{1}{\sin^2 \theta} \, d\theta = \int \csc^4 \theta \, d\theta - \int \csc^2 \, d\theta$

$= \int (\csc^2 \theta)(\csc^2 \theta) \, d\theta - \int \csc^2 \theta \, d\theta = \int (1 + \cot^2 \theta) \csc^2 \theta - \int \csc^2 \theta \, d\theta$

$= \int \csc^2 \theta \, d\theta + \int \cot^2 \theta \csc^2 \theta \, d\theta - \int \csc^2 \theta \, d\theta = \int \cot^2 \theta \csc^2 \theta \, d\theta \longrightarrow$

$q = \cot\theta \quad dq = -\csc^2\theta\, d\theta$

$\overset{sub}{\Downarrow}$

$= \int \cot^2\theta \csc^2\theta\, d\theta = \int q^2 \csc^2\theta \dfrac{dq}{-\csc^2\theta} = -\int q^2 dq = -\tfrac{1}{3}q^3 = -\tfrac{1}{3}\cot^3\theta$

to find $\cot\theta$, draw picture

$= -\tfrac{1}{3}\left(\dfrac{\sqrt{1-x^2}}{x}\right)^3 + C = -\dfrac{1}{3x}(1-x^2)^{3/2} + C$

triangle: 1, x, $\sqrt{1-x^2}$, $\quad \cot\theta = \dfrac{adj}{opp} = \dfrac{\sqrt{1-x^2}}{x}$

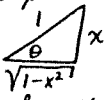14) $\int \sqrt{1-x^2}\, dx$ trig sub $a=1, x=u \quad x=\sin\theta, dx= \cos\theta\, d\theta$

sub $\int\sqrt{1-\sin^2\theta}\cos\theta\, d\theta = \int\cos^2\theta\, d\theta = \int(\tfrac{1}{2}+\tfrac{1}{2}\cos2\theta)d\theta$

sub in $\cos^2\theta = \tfrac{1}{2}+\tfrac{1}{2}\cos2\theta$

$= \tfrac{1}{2}\int d\theta + \tfrac{1}{2}\int\cos2\theta\, d\theta = \tfrac{1}{2}\theta + \tfrac{1}{4}(2\sin2\theta) = \tfrac{1}{2}\theta + \sin2\theta + C$

Now have to put into terms of x.

draw picture: triangle $1, x, \sqrt{1-x^2}$

$\theta = \arcsin x$

Since you can put $\sin2\theta = 2\sin\theta\cos\theta$

$\sin2\theta = (2)(x)(\sqrt{1-x^2}) = 2x\sqrt{1-x^2}$

So the answer is $\dfrac{\arcsin x + 2x\sqrt{1-x^2}}{2} + C$

---

15) $\int\dfrac{dx}{x^2-4} = \int\dfrac{dx}{(x-2)(x+2)}$ By partial fractions - find the fractions

$\dfrac{A}{(x-2)} + \dfrac{B}{(x+2)}$

$A(x+2) + B(x-2) = 1$
$Ax+2A+Bx-2B = 1$
$A+B=0 \quad A=-B$
$2A-2B=1$
$4A=1 \quad A=\tfrac{1}{4}, B=-\tfrac{1}{4}$

$\int\left(\dfrac{\tfrac{1}{4}}{(x-2)} + \dfrac{(-\tfrac{1}{4})}{(x+2)}\right)dx$

$= \tfrac{1}{4}\int\dfrac{1}{(x-2)}dx - \tfrac{1}{4}\int\dfrac{1}{x+2}dx$

$= \tfrac{1}{4}\log|x-2| - \tfrac{1}{4}\log|x+2| + C$

$= \tfrac{1}{4}\log\left|\dfrac{x-2}{x+2}\right| + C$

---

16) $\int\dfrac{2x+3}{x(x-1)^2}dx$ find the fractions - partial fraction

$\dfrac{A}{x} + \dfrac{B}{(x-1)} + \dfrac{C}{(x-1)^2}$

$A(x-1)^2 + Bx(x-1) + C(x) = 2x^2+3$
$Ax^2 - 2Ax + A + Bx^2 - Bx + Cx$
$A+B=2 \implies 3+B=2, B=-1$
$-2A-B+C=0 \implies -6+1+C=0, C=5$
$A=3$

$\int\tfrac{3}{x}dx + \int\dfrac{-1\,dx}{(x-1)} + \int\dfrac{5\,dx}{(x-1)^2} = 3\log|x| - \log|x-1| - 5\left(\dfrac{1}{x-1}\right) + C$

$u=(x-1)$
$du=dx$
$\int\dfrac{5\,du}{u^2} = -5u^{-1}$

$u = \log x \qquad dv = dx$
$du = \tfrac{1}{x}dx \qquad v = x$

17) $\int\log x\, dx$ · By parts

$= x\log x - \int\tfrac{1}{x}x\, dx = x\log x - \int dx = x\log x - x + C$

18) $\int\dfrac{xe^x}{(x+1)^2}dx$ By parts $\quad u=xe^x \qquad dv = \dfrac{1}{(x+1)^2}dx$
$du = (x+1)e^x dx \qquad v = -\dfrac{1}{x+1}$

$uv - \int v\, du$

$= xe^x\left(-\dfrac{1}{x+1}\right) - \int -\dfrac{1}{x+1}(x+1)e^x dx$

$= \dfrac{-xe^x}{x+1} + \int e^x dx = \dfrac{-xe^x}{x+1} + e^x + C$

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19) $\int x\sqrt{x+1}\, dx$ BY PARTS $\quad u=x \qquad dv = \sqrt{x+1}\, dx$
$du = dx \qquad v = \tfrac{2}{3}(x+1)^{3/2}$

$= x\left(\tfrac{2}{3}(x+1)^{3/2}\right) - \int\tfrac{2}{3}(x+1)^{3/2}dx$

$= \tfrac{2}{3}x(x+1)^{3/2} - \tfrac{2}{3}\left(\tfrac{2}{5}\right)(x+1)^{5/2} + C = \tfrac{2}{3}x(x+1)^{3/2} - \tfrac{4}{15}(x+1)^{5/2} + C$

---

20) $\int\dfrac{dx}{1+2\cos x}$ use $z=\tan\tfrac{x}{2}$ substitution
$\cos x = \dfrac{1-z^2}{1+z^2} \quad dx = \dfrac{2\,dz}{1+z^2}$

substitute in $\int\dfrac{\tfrac{2\,dz}{1+z^2}}{1+2\left(\tfrac{1-z^2}{1+z^2}\right)} = \int\dfrac{2\,dz}{(1+z^2)+2(1-z^2)} = \int\dfrac{2\,dz}{1+z^2+2-2z^2} = \int\dfrac{2\,dz}{-z^2+3}$

$= \int\dfrac{-2\,dz}{z^2-3}$

Next use partial fractions

$\int\left(\dfrac{A}{z+\sqrt{3}} + \dfrac{B}{z-\sqrt{3}}\right)dz$

$(A(z-\sqrt{3})+B(z+\sqrt{3})) = -2$
$Az-A\sqrt{3}+Bz+B\sqrt{3} = -2$
$A+B=0 \Rightarrow A=-B$
$-A+B=-\tfrac{2}{\sqrt{3}}$
$A=+\tfrac{1}{\sqrt{3}}, B=-\tfrac{1}{\sqrt{3}}$

$= \int\left(\dfrac{+\tfrac{1}{\sqrt{3}}}{z+\sqrt{3}} + \dfrac{-\tfrac{1}{\sqrt{3}}}{z-\sqrt{3}}\right)dz$

$+ \tfrac{1}{\sqrt{3}}\log|z+\sqrt{3}| - \tfrac{1}{\sqrt{3}}\log|z-\sqrt{3}| = \tfrac{1}{\sqrt{3}}\log\left|\dfrac{z+\sqrt{3}}{z-\sqrt{3}}\right| + C$

$= \tfrac{1}{\sqrt{3}}\log\left|\dfrac{\tan\tfrac{x}{2}+\sqrt{3}}{\tan\tfrac{x}{2}-\sqrt{3}}\right| + C$

---

21) $\int\dfrac{\sec x}{2\tan x+\sec x-1}dx = \int\dfrac{\tfrac{1}{\cos x}}{2\left(\tfrac{\sin x}{\cos x}\right)+\tfrac{1}{\cos x}-1}dx = \int\dfrac{1}{2\sin x+1-\cos x}dx$ $\quad\{$ sub in $\cos x = \dfrac{1-z^2}{1+z^2}$

$\sin x = \dfrac{2z}{1+z^2}$

$dx = \dfrac{2\,dz}{1+z^2}$

sub in $= \int\dfrac{\tfrac{2\,dz}{1+z^2}}{2\left(\tfrac{2z}{1+z^2}\right)+1-\left(\tfrac{1-z^2}{1+z^2}\right)} = \int\dfrac{2\,dz}{4z+(1+z^2)-(1-z^2)}$

$= \int\dfrac{2\,dz}{4z+2z^2} = \int\dfrac{dz}{2z+z^2}$ use partial fractions for $\int\left(\dfrac{A}{z}+\dfrac{B}{2+z}\right)dz$

$A(2+z)+Bz = 1$
$A+B=0 \quad 2A=1, A=\tfrac{1}{2}$
$B=-\tfrac{1}{2}$

$$\int\left(\frac{\frac{1}{2}}{z} + \frac{-\frac{1}{2}}{z+2}\right)dz = \frac{1}{2}\log|z| - \frac{1}{2}\log|z+2| = \frac{1}{2}\log\left|\frac{z}{z+2}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{\tan\frac{x}{2}}{\tan\frac{x}{2}+2}\right| + C$$

---

22) $\int\frac{dx}{1+\sqrt{x}}$ $u = 1+\sqrt{x}$ $du = \frac{1}{2}x^{-\frac{1}{2}}dx$, $2\sqrt{x}\,du = dx$

sub in $\Rightarrow \int\frac{2\sqrt{x}\,du}{u}$ and $\sqrt{x} = u-1$

$$\Rightarrow \int\frac{2(u-1)du}{u} = \int 2\,du - \int\frac{2\,du}{u} = 2u - 2\log|u|$$

$$= 2(1+\sqrt{x}) - 2\log|1+\sqrt{x}| + C$$

---

23) $\int\sqrt{1-e^x}\,dx$ $u^2 = 1-e^x \leftarrow$ solve for x

$-u^2+1 = e^x$, $\ln|-u^2+1| = x$

$$= \int u\left(\frac{-2u}{1-u^2}\right)du \qquad \frac{-2u}{1-u^2}du = dx$$

$$= \int\frac{-2u^2}{1-u^2}\,du \quad \text{Divide numerator by denominator}$$

$$= \int\left(2 - \frac{2}{-u^2+1}\right)du = \int 2\,du + \int\frac{2}{u^2-1}\,du$$

partial fractions

$$= \int 2\,du + \int\left(\frac{1}{u-1} + \frac{-1}{u+1}\right)du$$

$\frac{A}{u-1} + \frac{B}{u+1}$

$A(u+1) + B(u-1) = 2$
$A+B = 0 \Rightarrow A = -B$
$A-B = 2$, $A = 1$
$-2B = 2$
$B = -1$

$$= 2u + \log|u-1| - \log|u+1|$$

$$= 2\sqrt{1-e^x} + \log\left|\frac{\sqrt{1-e^x}-1}{\sqrt{1+e^x}+1}\right| + C$$

---

24) $\int\frac{dx}{2x^2+8x+20} = \frac{1}{2}\int\frac{dx}{x^2+4x+10} = \frac{1}{2}\int\frac{dx}{(x^2+4x+4)+10-4} = \frac{1}{2}\int\frac{dx}{(x+2)^2+6}$

complete square

Sub $u = x+2$ $\frac{1}{2}\int\frac{du}{u^2+6} = \frac{1}{2}\int\frac{du}{(\frac{u}{\sqrt{6}})^2+1} = \frac{1}{2}\left(\frac{1}{\sqrt{6}}\right)\arctan\left(\frac{u}{\sqrt{6}}\right)$
$du = dx$

$$= \frac{1}{2\sqrt{6}}\arctan\left(\frac{x+2}{\sqrt{6}}\right) + C$$

---

25) $\int\frac{dx}{\sqrt{x^2+x+1}} \xleftarrow{\text{complete the square}} = \int\frac{dx}{\sqrt{(x^2+x+\frac{1}{4})+1-\frac{1}{4}}} = \int\frac{dx}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}$

Now trig substitutions

$\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}} = \sqrt{\frac{3}{4}\tan^2\theta+\frac{3}{4}} = $
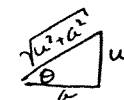
$\frac{\sqrt{3}}{2}\sqrt{\tan^2\theta+1} = \frac{\sqrt{3}}{2}\sec\theta$

$a = \frac{\sqrt{3}}{2}$ $u = x+\frac{1}{2}$
$u = a\tan\theta$
$x+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$
$dx = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$

$$\int\frac{\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta}{\frac{\sqrt{3}}{2}\sec\theta} = \int\sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}{\frac{3}{4}} + \frac{x+\frac{1}{2}}{\frac{3}{4}}\right| + C = \ln\left|\frac{\sqrt{x^2+x+1}}{3/4} + \frac{x+\frac{1}{2}}{3/4}\right| + C$$