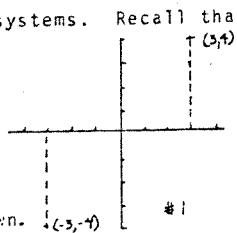**I. Why polar coordinates?** What advantages do polar coordinates have over rectangular coordinates? For one thing, polar coordinates often lead to easier compuations than rectangular coordinates would for a given problem. This alone makes it worthwhile to know how to work with polar coordinates. More importantly, though, polar coordinates arise quite naturally in many applications -- honey bees use a polar coordinate system to tell other members of their hive how to get to a newly discovered food source, birds use a polar coordinate system to help them in their migrations, and the equation for the exit ramps on most highways can best be expressed in polar coordinates. Since polar coordinates come up in problems from fields as diverse as biology and civil engineering, its probably worth your time to learn something about them.

II. What are polar coordinates? The Cartesian (or rectangular) coordinate system (named after Rene Descartes (1596-1650)) has been so successful since its introduction that it is easy to forget that there are other equally valid coordinate systems. Recall that to graph the point (3,4) in the Cartesian system, one moves 3 units to the right (from a fixed origin) and then 4 units up. Negative numbers are taken care of by moving to the left and down.
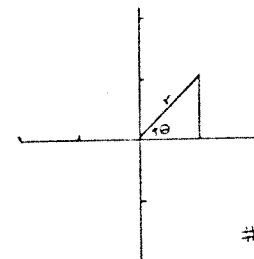
Polar coordinates work slightly differently - one finds the directed distance, r, from the origin to the point, and the directed angle, $\thet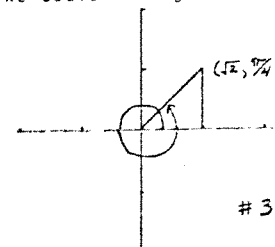a$, between the x-axis and the line segment connecting the point and the origin. The point is then referred to as (r,θ). An example should help clarify what all this means.

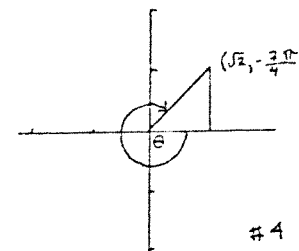Example #1. Find the polar coordinates for the point (1,1). r is the distance from the origin to the point, so $r = \sqrt{(1-0)^2+(1-0)^2} = \sqrt{1^2+1^2} = \sqrt{2}$. θ is the angle shown in graph #2. We know (from basic trigonometry) that $\tan\theta = \frac{y}{x} = \frac{1}{1}$, so $\theta = \arctan(1) = \frac{\pi}{4}$. The polar coordinates for the point are $(r,\theta) = (\sqrt{2},\frac{\pi}{4})$.

Unfortunately, we are not done yet. We could also get to the point $(\sqrt{2},\frac{\pi}{4})$ by letting θ go around in a circle once (or any number of times) before coming to rest (see graph #3). In this case, the polar coordinates for the point woul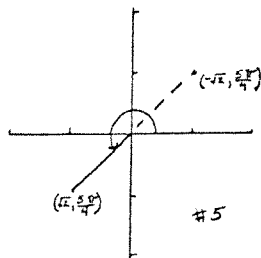d be $(\sqrt{2},\frac{\pi}{4}+2\pi n)$ where n is the number of times θ goes around before stopping. We could also let θ go in the opposite direction (see graph #4) to find that $(\sqrt{2},\frac{\pi}{4})$ can also be written as $(\sqrt{2},\frac{-7\pi}{4})$. And once again, θ is allowed to circle around as many times as it wants, so we get $(\sqrt{2},\frac{-7\pi}{4}-2\pi n)$ as another way to write the point in polar coordinates.

There is one more small hitch (see graph #5).

If we let $\theta = \frac{5\pi}{4}$, and then go in the opposite direction, we can end up at our point once again - only this time we call it $(-\sqrt{2},\frac{5\pi}{4})$ instead of $(\sqrt{2},\frac{\pi}{4})$. Of course, $(-\sqrt{2},\frac{5\pi}{4}+2\pi n)$ and $(-\sqrt{2},\frac{-3\pi}{4}-2\pi n)$ are also possibilities.

One thing should be very clear from this example - any point can be represented in many different ways in polar coordinates. This will lead to some interesting situations - as in Example #2.

Example #2. Show that $(3,\frac{3\pi}{4})$ is on the curve $r = 3 \sin 2\theta$. We first attempt to show this by letting $r = 3$, $\theta = \frac{3\pi}{4}$, so $r = 3 \sin 2\theta$ becomes $3 = 3 \sin 2(\frac{3\pi}{4}) = 3 \sin(\frac{3\pi}{2}) = -3$, which certainly isn't true!! However, we note that $(3,\frac{3\pi}{4})$ can also be written as $(-3,\frac{-\pi}{4})$. Then $r = 3 \sin 2\theta$ becomes $-3 = 3 \sin(2)(\frac{-\pi}{4})$ or $-3 = 3 \sin(\frac{-\pi}{2}) = -3$, which is true. So $(-3,\frac{-\pi}{4})$ is on the curve $r = 3 \sin 2\theta$ since it satifies the equation of the curve, which means $(3,\frac{3\pi}{4})$ is on the curve (after all, $(3,\frac{3\pi}{4})$ is just another name for $(-3,\frac{-\pi}{4})$). This means we have found a point - $(3,\frac{3\pi}{4})$ - which doesn't satisfy the equation directly, but which does satisfy the equation when rewritten as $(-3,\frac{-\pi}{4})$.

We hope these examples don't frighten you too much - we just want you to understand that polar coordinates have some pecularities one doesn't find in cartesian coordinates. We now turn to some easier material.

III. <u>Converting to polar coordinates</u>. It is sometimes necessary to convert an equation given in rectangular coordinates into one in polar coordinates. Fortunately, this is a very easy process; we just use the substitutions:

$$x = r \cos \theta$$
$$y = r \sin \theta$$
$$\frac{y}{x} = \tan \theta$$

Example #3. Convert x^2+y^2 into polar coordinates. (This is a very easy example, but it is quite useful.)

$$x^2+y^2 = (r \cos \theta)^2 + (r \sin \theta)^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2(\cos^2\theta + \sin^2\theta) = r^2 .$$

Example #4. Convert $x^2+y^2 = 4x$ into polar coordinates.

$$r^2 = 4 r \cos \theta \quad \text{(see example #3)}$$
$$r = 4 \cos \theta$$

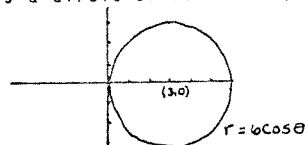Example #5. Convert $x = y$ into polar coordinates.
$$x = y$$
$$r \cos \theta = r \sin \theta$$
$$\cos \theta = \sin \theta$$
$$1 = \tan \theta$$
$$\theta = \tan^{-1}(1), \text{ i.e., } \theta = \frac{\pi}{4}$$

IV. <u>Converting to rectangular coordinates</u>. Just as we sometimes need to convert from rectangular coordinates, we also sometimes need to reverse the process and convert from polar coordinates into rectangular coordinates. Unfortunately, it is usually a bit trickier to move in this direction. However, there are some standard techniques, which we will demonstrate with a few examples.

Example #6. Convert $r = 6 \cos \theta$ into rectangular coordinates, and then graph the function.

Multiplying both sides of the equation by r, we get $r^2 = 6r \cos \theta$, or $x^2 + y^2 = 6x$. Completing the square, we get $x^2 - 6x + 9 + y^2 = 9$ or $(x-3)^2 + y^2 = 9$. This is a circle of radius 3, with center at $(3,0)$.

Example #7. Convert $r = \cos 2\theta$ into rectangular coordinates. Note that $r \cos \theta = x$, but $r \cos 2\theta \neq 2x$. Instead, we need to use a trig identity: $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$.
Then $r = \cos 2\theta = \cos^2 \theta - \sin^2 \theta$. Multiplying both sides by r^2, we get $r^3 = r^2 \cos^2 \theta - r^2 \sin^2 \theta$, or $r^3 = x^2 - y^2$. Now $r^3 = (r^2)(r) = (x^2 + y^2)(\sqrt{x^2 + y^2})$, so we end up with $(x^2 + y^2)(\sqrt{x^2 + y^2}) = x^2 - y^2$; or $(x^2 + y^2)^{3/2} = x^2 - y^2$ (which should make you appreciate the simplicity of the equation in polar form).
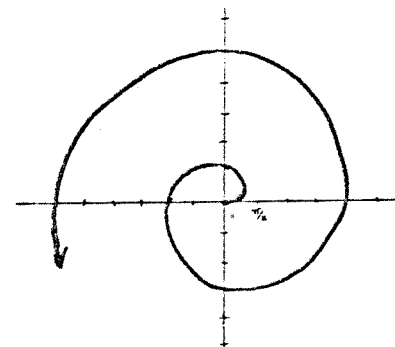
V. Plotting the Points. Unfortunately, converting equations from polar coordinates into rectangular coordinates is a help in graphing only in fairly simple cases. There are many equations where it is difficult (or impossible) to convert to an equivalent "rectangular" equation, or where the rectangular form of the equation is so complicated that it only makes matters worse. An example of an equation of this type is $r = \theta$ (you should spend a minute or two trying to convert this into an equation in rectangular coordinates - but don't spend too much longer than that!).

So what can you do when you run into an equation of this type? You can go back to the old "tried and true" method of plotting points until you can "guess" at what the graph looks like. And, in addition

to plotting points, you can use the methods of graphing you learned for equations in rectangular form (such as using first derivatives as an aid in graphing).

Example #8. $r = \theta$ (The Archimedean spiral - so named because Archimedes (287-212 BC) was the first person to define/discover this curve.)

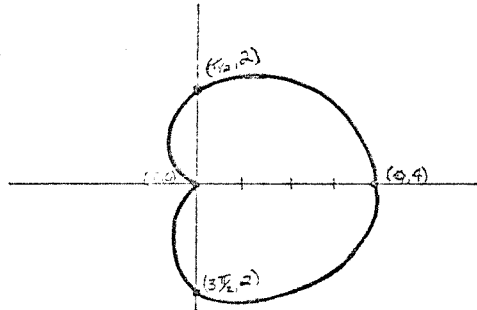| r	θ
0	0
$\frac{\pi}{3}$	$\frac{\pi}{3}$
$\frac{\pi}{4}$	$\frac{\pi}{4}$
$-\frac{\pi}{2}$	$-\frac{\pi}{2}$
$-\frac{3\pi}{2}$	$-\frac{3\pi}{2}$

Now we use fact $(-r, \theta) = (r, \theta + \pi)$, so $(-\frac{\pi}{2}, -\frac{\pi}{2}) = (\frac{\pi}{2}, \frac{\pi}{2})$. A little thought will convince you that plotting negative values of θ will not give us any points on the graph that we couldn't get by graphing positive values of θ.

We can also use techniques from calculus to help us graph this function. Since $r = \theta$, $\frac{dr}{d\theta} = 1$. This means that as θ grows larger, r also grows larger.

The last page of this capsule is a piece of polar graph paper which you may slide under your paper as a guide for more accurate drawing.

Example #9. $r = 2(1 + \cos\theta)$ Using calculus, note $\frac{dr}{d\theta} = -2\sin\theta$. This tells us that between $\theta = 0$ and $\theta = \pi$, $\frac{dr}{d\theta}$ is negative (since $\sin\theta$ is positive in this range); while between $\theta = \pi$ and $\theta = 2\pi$, $\frac{dr}{d\theta}$ is positive. This means that r is getting closer to the origin as θ goes from 0 to π, and gets further from the origin as θ goes from π to 2π. We can deduce from this that r reaches its largest value when $\theta = 0$ and its smallest value when $\theta = \pi$. (Note that $\theta = 0$ and $\theta = \pi$ are the two critical points of the function, since $\frac{dr}{d\theta} = 0$ at these points). We now compute a few points (and use the above information to connect the points) to get the following graph:

θ	r
0	4
$\frac{\pi}{2}$	2
π	0
$\frac{3\pi}{2}$	2

VI. Additional hints for graphing in polar coordinates.

When graphing a function, it is always nice to know whether or not the graph is symmetric about any line, especially whether or not it is symmetric about an axis. For, if a graph is symmetric about an axis, you only need to figure out what the function looks like on one side of the axis, then use the symmetry to find out what the function looks like on the other side of the axis.

The following table is useful in locating symmetries in polar coordinates - if the substitution does not change the solutions to the equation, then the graph has the given symmetry:

Substitution	Symmetry
$-\theta$ for θ	x-axis
$-r$ for r	origin
$\pi-\theta$ for θ	y-axis

Example #9. Find which of the following curves are symmetric about the origin, the x-axis and/or the y-axis.

a) $r = 2\cos\theta$

b) $r^2 = 4$

c) $r = 2 + 2\sin\theta$

a) We try the substitutions: First try $-\theta$ for θ. $r = 2\cos(-\theta)$. Since $\cos(-\theta) = \cos\theta$, the equation isn't changed by this substitution. This means the curve is symmetric with respect to the x-axis.

Next, try $-r$ for r. This gives us $-r = 2\cos\theta$, which has $(-2,0)$ as a solution. Since $(-2,0)$ is not a solution to the original equation, the curve is not symmetric with respect to the origin.

We also can try $(\pi-\theta)$ for θ, getting $r = 2\cos(\pi-\theta)$, or $r = -2\cos\theta$. This has $(-2,0)$ as a solution, and $(-2,0)$ doesn't satisfy the original equation, so the curve is not symmetric to the y-axis.

b) $r^2 = 4$. Note that using $-r$ for r gives us $(-r)^2 = 4$ or $r^2 = 4$, so the curve is symmetric with respect to the origin. Also, the substitutions $-\theta$ for θ and $\pi-\theta$ for θ don't change anything (since there are no θ's in equation!!) so the curve is symmetric with respect to both the x-axis and the y-axis.

c) The reader should make the necessary substitutions to find that $r = 2 + 2\sin\theta$ is symmetric with respect to the y-axis, but not with respect to the x-axis or the origin.

Problems

1. The following points are given in polar coordinates. Graph the points, and give all possible polar coordinates for each point.

   a) $(2, \frac{\pi}{4})$

   b) $(-3, \frac{\pi}{6})$

   c) $(-3, \frac{-\pi}{6})$

2. a) Show that $(2, \frac{3}{4}\pi)$ is on the curve $r = 2 \sin 2\theta$

   b) Show that $(\frac{1}{2}, \frac{3}{2}\pi)$ is on the curve $r = -\sin(\frac{\theta}{3})$

3. Convert the following to equivalent equations in polar coordinates:

   a) $x^2 + y^2 = 4$

   b) $(x-2)^2 = 9 - y^2$

   c) $x = y$

   d) $3x + 2y = 5$

   e) $y^2 + 2y = 3 - x^2$

4. Convert the following to equivalent equations in x and y, and use this to help you sketch the graph of the polar equations.

   a) $r(\sin\theta + 3 \cos \theta) = 2$

   b) $r = \sec \theta$

   c) $r = 2 \tan \theta$

   d) $\theta = \frac{\pi}{3}$ (hint: see problem 3c)

   e) $r = 4 \sin \theta$

5. Graph the following:

   a) $r = 3$

   b) $r = -3$

   c) $\theta = \frac{-\pi}{6}$

   d) $r = 3 \cos 2\theta$

   e) $r = 2 \cos 3\theta$

   f) $r = 5 \sin \theta$

   g) $r = 2 \sin 4\theta$

   h) $r^2 = 2 \cos 2\theta$

   i) $r = 2(1 + \cos \theta)$

   j) $r = -3(1 - \sin \theta)$

Solutions.

1. a)

$(2, \frac{\pi}{4}) = (2, \frac{\pi}{4} + 2\pi n)$

or $= (-2, \frac{3\pi}{4} + 2\pi n) [n = 0, \pm 1, \pm 2 \ldots]$

   b)

$(-3, \frac{\pi}{6}) = (-3, \frac{\pi}{6} + 2\pi n)$

or $= (3, \frac{7\pi}{6} + 2\pi n) [n = 0, \pm 1, \pm 2 \ldots]$

   c)

$(-3, \frac{-\pi}{6}) = (-3, \frac{-\pi}{6} + 2\pi n)$

or $= (3, \frac{7\pi}{6} + 2\pi n) [n = 0, \pm 1, \pm 2 \ldots]$

2. a) If we plug in $r = 2$, $\theta = \frac{3}{4}\pi$, we get $2 = 2 \sin 2(\frac{3\pi}{4})$ or $2 = -2$ which isn't true. However $(2, \frac{3}{4}\pi)$ can also be written as $(-2, \frac{-\pi}{4})$. If we use $r = -2$, $\theta = \frac{-\pi}{4}$ we get $-2 = 2 \sin 2(\frac{-3\pi}{4})$, or $-2 = -2$ (which is true). So $(-2, \frac{-\pi}{4})$ satisfies the equation, i.e., $(-2, \frac{-\pi}{4})$ is on the curve $r = 2 \sin 2\theta$. But $(-2, \frac{-\pi}{4}) = (2, \frac{3\pi}{4})$, so $(2, \frac{3\pi}{4})$ is on the curve.

   b) Note that $(\frac{1}{2}, \frac{3\pi}{2}) = (\frac{-1}{2}, \frac{\pi}{2})$, and follow the solution to part a.

3. a) $x^2 + y^2 = 4 \implies (r \cos \theta)^2 + (r \sin \theta)^2 = 4$; $r^2 \cos^2\theta + r^2 \sin^2\theta = 4$ $r^2(\cos^2\theta + \sin^2\theta) = 4$ $r^2 = 4$ $r = 2$

   b) $(x-3)^2 = 9 - y^2$ $x^2 - 4x + 9 = 9 - y^2$ $x^2 + y^2 = 4x$ $r^2 = 4r \cos \theta$ $r = 4 \cos \theta$

c) $x = y$ $r \cos \theta = r \sin \theta$ $\cos \theta = \sin \theta$ $1 = \tan \theta$

$\theta = \tan^{-1}(1)$ $\theta = + \frac{\pi}{4}$

d) $3x + 2y = 5$ $3r \cos \theta + 2r \sin \theta = 5$ $r(3 \cos \theta + 2 \sin \theta) = 5$

e) $y^2 + 2y = 3 - x^2$ $x^2 + y^2 = 3 - 2y$ $r^2 = 3 - 2r \sin \theta$

4. a) $r(\sin \theta + 3 \cos \theta) = 2$ $r \sin \theta + 3r \cos \theta = 2$

$y + 3x = 2$

$\leftarrow r(\sin \theta + 3\cos \theta) = 2$

b) $r = \sec \theta$ $r = \frac{1}{\cos \theta}$ $r \cos \theta = 1$

$x = 1$

$\leftarrow r = \sec \theta$

c) $r = \tan \theta$ $r = \frac{\sin \theta}{\cos \theta}$ $r \cos \theta = \sin \theta$ $r^2 \cos \theta = r \sin \theta$

$r(r \cos \theta) = r \sin \theta$ $\sqrt{x^2 + y^2}(x) = y$ $(x^2 + y^2)(x^2) = y^2$

$x^4 + x^2 y^2 = y^2$ $x^4 = y^2 - x^2 y^2$

$x^4 = y^2(1 - x^2)$ $\frac{x^4}{1 - x^2} = y^2$

$x = -1$ $x = 1$

d) $\theta = \frac{\pi}{3}$ $\theta = \tan^{-1}(\sqrt{3})$ $\tan \theta = \sqrt{3}$ $\frac{\sin \theta}{\cos \theta} = \sqrt{3}$ $\frac{r \sin \theta}{r \cos \theta} = \sqrt{3}$

$\frac{y}{x} = \sqrt{3}$ $y = \sqrt{3} \, x$

(to help understand how we "guessed"

to go from $\theta = \frac{\pi}{3}$ to $\theta = \tan^{-1}(\sqrt{3})$,
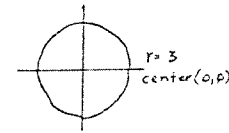
see problem 3c).

$\theta = \frac{\pi}{3}$

e) $r = 4 \sin \theta$ $r^2 = 4r \sin \theta$ $x^2 + y^2 = 4y$

$x^2 + y^2 - 4y = 0$ $x^2 + (y^2 - 4y + 4) = 4$

$x^2 + (y - 2)^2 = 4$

circle of radius 2, center at (0,2)
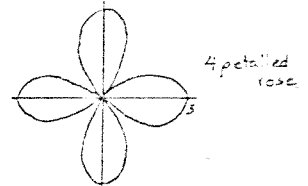
5.   a)

$r = 3$
center (0,0)

b)

$r = 3$
center = (0,0)
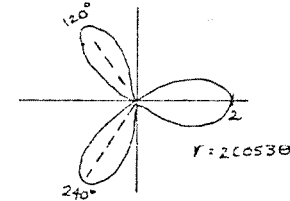
c)

$\theta = -\frac{\pi}{6}$

d)

4 petalled rose
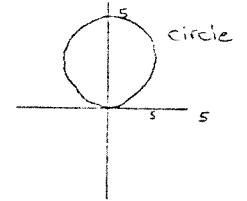
e) $r = 2 \cos 3\theta$

$120°$
$240°$
$r = 2 \cos 3\theta$

f) $r = 5 \sin \theta$

circle

g) $r = 2 \sin 4\theta$
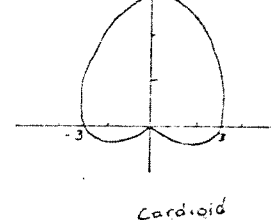
8 petalled rose
$r = 2$

h) $r^2 = 2 \cos 2\theta$

lemniscate
$r = \sqrt{2}$

i) $r = 2(1 + \cos \theta)$

4
cardioid

j) $r = -3(1 - \sin \theta)$

-3
cardioid