

Mathematics Support Capsules

DIFFERENTIAL EQUATIONS

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Notes on Differential Equations

§1. Differential Equations.

A differential equation is an equation having one or more derivatives or differentials. Examples:

$$(1) \frac{dy}{dx} = x \sin x ;$$

$$(2) \frac{dy}{dt} = y \sin t ;$$

$$(3) (x^2+y^2)dy - y^2dx = 0 ;$$

$$(4) y' + y = e^x ;$$

$$(5) \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 ;$$

$$(6) \frac{d^2 y}{dt^2} + y = 0 ;$$

$$(7) (y''')^2 + (y')^3 + \frac{y^2}{x+1} = \sin x \cos x .$$

Differential equations may be classified by

(a) order: the number of the highest derivative occurring in the equation (e.g., equation (4) has order 1, equation (6) above has order 2).

(b) type: ordinary differential equation or partial differential equation.

(c) degree: the exponent of the highest power of the highest-order derivative (e.g., (7) has degree 2, since the third derivative term has that power).

You have already seen some easy differential equations. For instance, in Chapter 9 of Thomas, you learned how to solve (i.e., integrate) equation (1) by means of a technique called integration by parts; the solution has the form

$$y(x) = -x \cos x + \sin x + C .$$

where C is an arbitrary constant. Note a few facts here:

- (a) a solution is a function $y(x)$ without derivatives such that it and its derivatives satisfy the given equation;
- (b) because C is an arbitrary constant, we do not get a single solution to a differential equation. Instead, we have what is sometimes called a family of solutions or envelope of solutions;
- (c) if we add a condition to our problem which fixes the arbitrary constant, we have added what is called a boundary condition or boundary value.

Example 1.1. As a result of leakage, an electrical condensor discharges at a rate proportional to the charge at any time t . If the charge Q has value 10 units at time $t=0$, find Q as a function of t .

Solution. The rate of change is $\frac{dQ}{dt}$. Since it is decreasing at a rate proportional to the charge itself, we have $\frac{dQ}{dt} = -kQ$. This is the differential equation we wish to solve. Note that it is an ordinary differential equation (type) of order 1 and degree 1.

Before we solve this equation, notice that we are given an extra piece of information, namely that $Q(0) = 10$. This is our boundary condition.

Now we wish to find a solution $Q = Q(t)$, i.e., Q at any time t . So

$$\frac{dQ}{dt} = -kQ \text{ gives}$$

$$\frac{dQ}{Q} = -k dt .$$

Thus $\ln Q = -kt + C$, so

$$Q = e^{-kt+C} = e^{-kt}e^C = Ae^{-kt} .$$

So at any time t ,

$$Q(t) = Ae^{-kt},$$

with A an arbitrary constant. Now we use our boundary value to find A for this problem.

Since $Q(0) = 10 = Ae^0 = A$, thus $Q(t) = 10e^{-kt}$.

We now show that $Q(t)$ satisfies the given differential equation. Since $Q(t) = 10e^{-kt}$, we must show that $\frac{dQ}{dt} = -10ke^{-kt}$. But this is clear.

Now that we've seen an example of a differential equation and its solution, let's look at some general kinds of ordinary differential equations which have solutions.

§2. Equations with variables separable.

Consider an ordinary differential equation which can be written in the form

$$f(y)dy = g(x)dx .$$

This kind of equation has all its y-terms with dy and all its x-terms with dx, and is thus called an equation with variables separable. It then has a solution

$$\int f(y)dy = \int g(x)dx + C .$$

Example 2.1. $\frac{dy}{dx} = y \sin x$ may be written

$$\frac{dy}{y} = \sin x \, dx ,$$

so $\ln y = -\cos x + C$ is a solution of this equation.

Exercise 2.

Find the equations below which have variables separable and solve them.

1. $y \ln y \, dx + (1+x^2)dy = 0$
2. $\frac{dy}{dx} = e^{y-x}$
3. $(x^2+y^2)dy - y^2dx = 0$
4. $\frac{dy}{dx} + 2y = e^{-x}$
5. $\sqrt{2xy} \, dy = dx$
6. $(xe^{y/x} + y)dx - xdy = 0$
7. $xdy + ydx = \sin x \, dx$
8. $(x+y)dx + (x+y^2)dy = 0$
9. $\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$
10. $(2xe^y + e^x)dx + (x^2+1)e^y dy = 0$
11. $(x-2y)dy + ydx = 0$
12. $y'' + w^2y = 0$, w a nonzero constant.
13. $x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$.

§3. Homogeneous Equations.

A differential equation which can be put into the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

is said to be homogeneous. It can then be solved by means of a substitution

$$v = \frac{y}{x}$$

which makes it a variables separable equation.

Example 3.1. $(x^2 + y^2)dx + 2xydy = 0$ is homogeneous, since

$$(a) \quad \frac{dy}{dx} = -\frac{(x^2 + y^2)}{2xy} = -\frac{1 + \left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}.$$

Letting $v = \frac{y}{x}$, $y = vx$;

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

So $v + x\frac{dv}{dx} = -\frac{1+v^2}{2v}$. Thus $\frac{dx}{x} + \frac{2v}{1+3v^2} dv = 0$, and

$$\ln x + \frac{1}{3} \ln(1+3v^2) = \frac{1}{3} \ln C.$$

Then

$$x^3(1+3v^2) = C,$$

$$x^3(1+3\left(\frac{y}{x}\right)^2) = C, \quad \text{and finally}$$

$$x(x^2 + 3y^2) = C.$$

Exercise 3. Find the equations in Exercise 2 which are homogeneous and solve them.

§4. First-order linear equations.

The linearity of a differential equation is a function of its dependent variable. To find out whether a given equation is linear, we compute the degree (not the order) of each of its terms, adding the exponents of the dependent variable, and of any of its derivatives that occur in the term.

If every term of a differential equation has degree zero or degree one, then the equation is said to be linear.

Example 4.1.

(a) $\frac{dy}{dx} + 3y = x$ is linear. The dependent variable is y .

$\frac{dy}{dx}$ has degree one, as does y . The independent variable x , has degree zero.

(b) $\frac{dy}{dx} + 3y^2 = x$ is not linear. Although $\frac{dy}{dx}$ has degree one, y^2 does not. Again x does not matter.

(c) $\frac{d^2y}{dx^2} + 3yx = \sin x$ is linear, but

(d) $\left(\frac{dy}{dx}\right)^2 + 3yx = \sin x$ is not linear.

(e) $\frac{d^ny}{dx^n} + by = 0$ is linear.

(f) $y \frac{dy}{dx} = x^2$ is not linear, since the left side of the equation has degree two (adding the degree of y and the degree of dy/dx).

(g) $x^2 \frac{dy}{dx} = y$ is linear. Each side has degree one.

Note that examples (c) and (e) are linear, but not first order (why?).

In this section we will only consider first order linear equations.

Every first order linear equation may be written in a standard

form:

$$(3F) \quad \frac{dy}{dx} + P(x)y = Q(x),$$

with $P(x)$ and $Q(x)$ functions of x . The solution of (SF) consists of finding a function $\rho(x)$ called an integrating factor, which turns (SF) into

$$(SF') \quad \rho(x) \frac{dy}{dx} + \rho(x)P(x)y = \rho(x)Q(x) = \frac{d}{dx}(\rho(x)y) .$$

The right side of this last gives our solution,

$$(S) \quad \rho(x)y = \int \rho(x)Q(x)dx + C .$$

So what is $\rho(x)$? From (SF'),

$$\rho(x) \frac{dy}{dx} + \rho(x)P(x)y = \rho(x) \frac{dy}{dx} + \left(\frac{d\rho}{dx}\right)y .$$

Thus

$$\rho(x)P(x) = \frac{d\rho}{dx}$$

gives

$$\frac{d\rho}{\rho(x)} = P(x)dx .$$

Hence

$$\ln \rho(x) = \int P(x)dx .$$

Finally,

$$\rho(x) = e^{\int P(x)dx}$$

is the integrating factor.

Although the above method looks hard, the same technique is used in all such problems, and becomes "trivial" by the third time it is used.

Example 4.2.

$\frac{dy}{dx} + y = e^x$ is first order linear, with $P(x) = 1$, $Q(x) = e^x$ in (SF). We need to find $\rho(x) = e^{\int P(x)dx}$.

$$\rho(x) = e^{\int dx} = e^x .$$

Our solution is then equation (S) on the previous page, $\rho(x)y = \int \rho(x)Q(x)dx + C$. So

$$e^x y = \int e^x \cdot e^x dx + C ,$$

$$e^x y = \int e^{2x} dx + C ,$$

$$e^x y = \frac{1}{2} e^{2x} + C ,$$

$$y = \frac{1}{2} e^x + C e^{-x} .$$

Example 4.3. $xdy = (3y + x^2)dx$ does not look first order linear, but it is, since it can be written as

$$\frac{dy}{dx} - \frac{3}{x} y = x .$$

Then $P(x) = -(3/x)$, $Q(x) = x$.

Thus

$$\rho(x) = e^{\int -(3/x)dx} = \frac{1}{x^3} .$$

So our solution is

$$\frac{y}{x^3} = \int \frac{x}{x^3} dx + C = -\frac{1}{x} + C .$$

Thus

$$y = -x^2 + Cx^3 .$$

Exercise 4. Find the first order linear equations in Exercise 2 and solve them. (Hint: there are three of them.)

§5. Exact differential equations.

Everyone remembers the total differential from Thomas; given $z = f(x, y)$, we can write $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$. For certain kinds of functions $f(x, y)$ (for instance, the continuous ones), the mixed second partials behave very nicely; namely,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

So if we look at an equation

$$(E) \quad M(x, y)dx + N(x, y)dy = 0,$$

it looks like our dz -function above, as long as $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. For this reason we say that if (E) satisfies $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, it is an exact differential equation. We can solve it by finding a function $z = f(x, y)$ such that

$$dz = dz = M(x, y)dx + N(x, y)dy = 0.$$

Example 5.1. $(xy^2 + y)dx + (x^2y + x)dy = 0$

is an exact equation, since, letting $M = xy^2 + y$ and $N = x^2y + x$, we have

$$\frac{\partial M}{\partial y} = 2xy + 1 = \frac{\partial N}{\partial x}.$$

Find a solution $f(x, y)$ as follows. We know $M = \frac{\partial f}{\partial x}$ by our first equation. So $f = \int M dx + C(y)$, where $C(y)$ is a function of y alone (since the x -derivative of a function of y is zero).

$$\begin{aligned} \text{Thus } f(x, y) &= \int (xy^2 + y)dx + C(y), \\ &= \frac{x^2 y^2}{2} + xy + C(y). \end{aligned}$$

Now we must find $C(y)$; but we know that $\frac{\partial f}{\partial y} = N$, so

$$x^2 y + x + C'(y) = x^2 y + x.$$

Thus $C'(y) = 0$, and $C(y) = C$, just an arbitrary constant. So our final solution is

$$f(x, y) = \frac{x^2 y^2}{2} + xy + C = 0$$

Example 5.2.

$$(e^x + \ln y + \frac{y}{x})dx + (\frac{x}{y} + \ln x + \sin y)dy = 0$$

is exact, since

$$\frac{\partial M}{\partial y} = \frac{1}{y} + \frac{1}{x} = \frac{\partial N}{\partial x}.$$

Our solution satisfies

$$\frac{\partial}{\partial x} f(x, y) = M, \text{ so}$$

$$\begin{aligned} f(x, y) &= \int M dx + C(y) \\ &= \int (e^x + \ln y + \frac{y}{x})dx + C(y) \\ &= e^x + x \ln y + y \ln x + C(y). \end{aligned}$$

Now $\frac{\partial f}{\partial y} = N$, so

$$\frac{x}{y} + \ln x + C'(y) = \frac{x}{y} + \ln x + \sin y.$$

$$\begin{aligned} \text{Thus } C'(y) &= \sin y, \text{ and} \\ C(y) &= -\cos y + C. \end{aligned}$$

Finally,

$$f(x, y) = e^x + x \ln y + y \ln x - \cos y + C = 0$$

There are other methods of solution for exact equations, including one which is sometimes referred to as "guesstimation",

which consists of continually asking oneself what more is necessary of $f(x,y)$ to get the M and N in the exact equation. Another such method involves finding an integrating factor (but not the same one as in §4).

Note: Many technical details as to what functions M and N actually have a solution $f(x,y)$ are skipped in the above. None of these details is trivial, and all involve such problems as whether M and N are continuous, whether they have partial derivatives, whether those partials are continuous, where M and N are defined, and more. Some of these questions are answered in Thomas, §15.13. These details are ignored in this course.

Exercise 5. Go back to Exercise 2; find the exact d.e.'s and solve them. (Hint: three more).

§6. Second order equations.

There are many; we study two types.

Type 1. $F(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0$. The dependent variable y is missing.

Solution: Reduce by a substitution $p = \frac{dy}{dx}$, $\frac{dp}{dx} = \frac{d^2y}{dx^2}$, to a first order equation

Example 6.1. $\frac{d^2y}{dx^2} = \sqrt{1 - (\frac{dy}{dx})^2}$.

This becomes $\frac{dp}{dx} = \sqrt{1 - p^2}$.

So $\frac{dp}{\sqrt{1-p^2}} = dx$,

$$\sin^{-1} p = x + C,$$

$$p = \sin(x+C).$$

But $p = \frac{dy}{dx}$, so $\frac{dy}{dx} = \sin(x+C)$, and

$$y = -\cos(x+C) + k.$$

Notice that we have actually integrated twice here, and that each time we have added an arbitrary constant. What does this say about a third order d.e.? How about an n^{th} order equation?

Type 2. $F(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0$. The independent variable x is missing.

Solution: Substitute $p = \frac{dy}{dx}$, $p \frac{dp}{dy} = \frac{d^2y}{dx^2}$.

Example 6.2. $\frac{d^2y}{dx^2} + y = 0$.

$$p \frac{dp}{dy} + y = 0, \text{ so}$$

$$p dp = -y dy.$$

Thus
$$\frac{p^2}{2} = \frac{y^2}{2} + \frac{k^2}{2}$$

$$p^2 = k^2 - y^2$$

$$\frac{dy}{dx} = p = \pm \sqrt{k^2 - y^2}$$

So
$$\frac{dy}{\sqrt{k^2 - y^2}} = \pm dx \quad \text{gives}$$

$$\sin^{-1} \frac{y}{k} = \pm x + A, \quad \text{and}$$

$$y = k \sin(A \pm x).$$

Exercise 6. You know (three of 'em).

§7. Homogeneous linear equations.

The idea of a linear equation was given in §4, where we discussed first order linear equations exclusively. Let us now consider higher order linear equations, e.g., those of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = F(x).$$

We will further choose each a_1 to be a constant (rather than as a function of x), and $F(x) = 0$ for now. When $F(x) = 0$, this kind of function is called a homogeneous linear equation of order n .

Example 7.1. $\frac{d^3 y}{dx^3} - 3 \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} = 0$ is a third order homogeneous linear equation. Solution of such equations is mainly high-school algebra, done as follows. Let $y = f(x)$. Then write $\frac{dy}{dx} = Df(x)$. D is called an operator on the function $f(x)$. Continuing with operator notation, we can write

$$D(Df(x)) = D^2 f(x) = \frac{d^2 y}{dx^2},$$

$$D(D(Df(x))) = D^3 f(x) = \frac{d^3 y}{dx^3}, \quad \text{etc.}$$

So the equation of the example above is, in operator notation,

$$(a) \quad D^3 f(x) - 3D^2 f(x) + 2Df(x) = 0.$$

This last gives

$$(D^3 - 3D^2 + 2D)f(x) = 0, \quad \text{so}$$

$$(D-2)(D-1)Df(x) = 0.$$

Now define $(\beta) \quad Df(x) = u,$

$$(\gamma) \quad (D-1)u = v,$$

$$(\delta) \quad (D-2)v = 0.$$

The equation for (6) is separable, and in the notation of §2, has the form

$$\frac{dv}{dx} = 2v.$$

Thus

$$v = Ce^{2x}.$$

Next substitute into (7);

$(D-1)u = Ce^{2x}$ is linear, and is written

$$\frac{du}{dx} - u = Ce^{2x}.$$

The solution to (7), by §4, is

$$e^{-x}u = \int e^x dx + D, \quad \text{or}$$

$$u = Ce^{2x} + De^x.$$

Substituting the last into (8),

$$\frac{dy}{dx} = Ce^{2x} + De^x, \quad \text{so}$$

$$y = Ce^{2x} + De^x + E,$$

C, D, E all constants. This, finally, is the solution.

So where, you ask, is the high school algebra? It's there, from equation (a). First see that (a) looks like the algebraic equation

$$r^3 - 3r^2 + 2r = 0.$$

Factoring, we get $(r-2)(r-1)r=0$, so roots of this equation are $r_1=2$, $r_2=1$, $r_3=0$. By the discussion following equations (6), (7), (8), we know we will have an exponential solution of the form

$$y = Ce^{r_1 x} + De^{r_2 x} + Ee^{r_3 x}.$$

This situation will always occur if (there's always an if) the roots of the algebraic equation are real and unequal.

Example 7.2. Solve $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.

Solution: The corresponding algebraic equation is

$$r^2 - 3r + 2 = 0,$$

with roots $r_1=2$, $r_2=1$, so the solution is

$$y = C_1 e^{2x} + C_2 e^x.$$

What if roots are not real and unequal? Two other cases exist: consider the second degree equation

$$ar^2 + br + c = 0.$$

Solutions always take the form

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

These solutions may be

- 1) real and unequal
- 2) real and equal (i.e., $r_1 = r_2$)
- 3) complex conjugates (i.e., $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$).

Case 1 is done.

Case 2. If $r_1 = r_2 = r$, then our differential equation has the form

$$(D-r)^2 y = 0.$$

Replicating the early part of this section, call

$$\begin{aligned} \text{(a)} \quad & (D-r)y = u \\ \text{(b)} \quad & (D-r)u = 0. \end{aligned}$$

Solve (b) by §2, getting

$$u = Ce^{rx}.$$

Then (a) becomes

$$(D-r)y = Ce^{rx}, \quad \text{or}$$

$$\frac{dy}{dx} - ry = Ce^{rx}.$$

This is first order linear with integrating factor

$$\rho = e^{-rx}.$$

So

$$e^{-rx}y = C \int e^{rx} \cdot e^{-rx} dx + D.$$

Thus $e^{-rx}y = Cx + D$, and

$$(S_2) \quad y = Cxe^{rx} + De^{rx}.$$

Example 7.3. Solve $\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 4y = 0$.

Solution: Roots are $r_1 = r_2 = 2$, so $y = Cxe^{2x} + De^{2x}$ is our solution.

Case 3. If roots are complex conjugates, case 1 says that the solution should be

$$(S_3) \quad y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}, \quad \alpha, \beta \text{ reals.}$$

This is a perfectly good solution, but most textbooks use an equivalent one. A "famous formula" in mathematics is Euler's Formula:

$$e^{i\beta x} = \cos \beta x + i \sin \beta x$$

$$e^{-i\beta x} = \cos \beta x - i \sin \beta x.$$

Thus (S_3) becomes

$$y = Ce^{(\alpha+i\beta)x} + De^{(\alpha-i\beta)x}$$

$$y = e^{\alpha x} [(C+D) \cos \beta x + i(C-D) \sin \beta x]$$

$$(S'_3) \quad y = e^{\alpha x} [K_1 \cos \beta x + K_2 \sin \beta x].$$

Example 7.4. Solve $\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 2y = 0$.

Solution. $r^2 + 2r + 2 = 0$ has roots

$$r_1 = -1+i, \quad r_2 = -1-i.$$

So take $\alpha = -1$, $\beta = 1$ and get, from (S'_3) ,

$$y = e^{-x} [K_1 \cos x + K_2 \sin x].$$

Finally, you ask, what if we do not have a quadratic equation?

Example 7.5. Solve

$$(D+3)(D-2)^2(D+1)^4(D-1-i)(D-1+i)Dy = 0.$$

Solution: Roots are $r_1 = -3$ (one real root), $r_2 = r_3 = 2$ (real), $r_4 = r_5 = r_6 = r_7 = -4$ (real), $r_8 = 1+i$, $r_9 = 1-i$, $r_{10} = 0$. So the solution is

$$y = C_1 e^{-3x} + C_2 e^{2x} + C_3 x e^{2x} + C_4 e^{-4x} + C_5 x e^{-4x} + C_6 x^2 e^{-4x} + C_7 x^3 e^{-4x} + e^x [C_8 \sin x + C_9 \cos x] + C_{10}.$$

O.K? Believe it? It is high school algebra. (So far.)

Exercise 7. Solve the equations below.

1. $\frac{d^2 y}{dt^2} - y = 0$
2. $\frac{d^2 y}{dt^2} + y = 0$
3. $y'' + 6y' + 5y = 0$
4. $y'' - 10y' + 16y = 0$
5. $\frac{d^4 y}{dx^4} - 4\frac{d^2 y}{dx^2} + 4y = 0$
6. $\frac{d^3 y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$
7. $(D-6)^4(D+2-i)(D+2+i)(D-\sqrt{2})Dy = 0$
8. Use differential equations methods to show that

$$(D-4)^4 y = 0$$

has the solution

$$y = C_1 e^{4x} + C_2 x e^{4x} + C_3 x^2 e^{4x} + C_4 x^3 e^{4x}.$$

9. Given $e^{i\beta x} = \cos \beta x + i \sin \beta x$, prove

$$e^{-i\beta x} = \cos \beta x - i \sin \beta x.$$

§8. Nonhomogeneous equations.

A linear equation has the form

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = F(x).$$

If $F(x) \neq 0$, the equation is nonhomogeneous.

Example 8.1. $\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = 0$ is homogeneous,

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = \sin x \text{ is nonhomogeneous.}$$

Good books in differential equations would now teach you how to solve nonhomogeneous equations. We won't do that. What we will do is show you one example.

Example 8.2. Solve: (A) $\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = \sin x$.

Solution. The corresponding homogeneous equation is $(D^2 + 2D + 1)y = 0$, having solution

$$y_h = C_1 e^{-x} + C_2 x e^{-x}.$$

This is our solution here also, or rather part of our solution (this part is called the general solution of the homogeneous case, which is why we write y_h here). What we also need is a particular solution y_p such that if we take two derivatives of y_p and put them into the left side of (A), we will get $\sin x$. Well, $\sin x$ can only have derivatives $A \sin x$ and $B \cos x$, A, B constants. So assume a particular solution of the form

$$y_p = A \sin x + B \cos x.$$

Then

$$y_p' = A \cos x - B \sin x$$

$$y_p'' = -A \sin x - B \cos x.$$

so

$$-A \sin x - B \cos x + 2A \cos x - 2B \sin x + A \sin x + B \cos x = \sin x,$$

$$\text{and } 2A \cos x = 0 \cos x = 0, \text{ so } A = 0$$

$$-2B \sin x = \sin x, \text{ so } B = -\frac{1}{2}.$$

$$\text{Finally, } y_p = -\frac{1}{2} \cos x.$$

Our solution is then

$$y = y_h + y_p = C_1 e^{-x} + C_2 x e^{-x} - \frac{1}{2} \cos x.$$

We unashamedly give no proof that adding y_h to y_p gives the most general solution.

Exercise 8.

1. By "working backwards", show that

$$y = C_1 e^{-x} + C_2 x e^{-x} - \frac{1}{2} \cos x$$

satisfies equation (A) of Example 8.2

2. Use the techniques of this section to solve

$$(a) y'' + 2y' - 3y = 6x + 2,$$

$$(b) y'' - y = e^{2x}.$$

3. Solve $y'' - y = e^x$.

This ends our discussion of general techniques.

§9. Some harder questions.

Question 1. Which of the differential equations in §1 can we now solve? By what techniques? (Comment: The others will be solvable at some later date in the 293-294 sequence, either by means of infinite series (Chapter 3 of the text), or by Fourier series (Chapter 8)).

Question 2. In §3 we studied

$$(E) \quad M dx + N dy = 0.$$

Find the differential equation which represents all the curves perpendicular to (E). Then find the family of solutions of the differential equation

$$2xy dy + (x^2 - y^2) dx = 0,$$

and of the differential equation representing all the curves perpendicular to the last equation.

Question 3. In example 1.1, we first found a general solution

$$Q(t) = A e^{-kt}$$

for the equation $\frac{dQ}{dt} = -kQ$.

- (a) Letting $k = 1$, use various numbers for A to graph the envelope of solutions represented by the equation $Q(t) = A e^{-t}$.
- (b) Now find the graph of the single solution which is specified by the boundary value.

Question 4. A body of mass m is suspended from a spring. The body is pulled from rest by an amount A and released. Use Newton's law, $F = ma$, and Hooke's law, which says that tension

in a spring is proportional to the amount stretched x , where k is the spring constant.

- (a) Find a differential equation for the motion.
(b) Solve this differential equation.

Question 5. An electrical circuit has a capacitor of capacity C farads, a coil of inductance L henrys, a resistance of R ohms, and a generator which produces an e.m.f. of E volts, in series. If current intensity at time t at some point of the circuit is I amperes, the differential equation for the current I is

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = \frac{dE}{dt}.$$

Find I as a function of time if

- (a) $R = 0$, $1/LC = \omega^2$, $E = \text{constant}$.
(b) $R = 0$, $1/LC = \omega^2$, $E = A \sin \omega t$.
(c) $R = 0$, $1/LC = \omega^2$, $E = A \sin \omega t$.

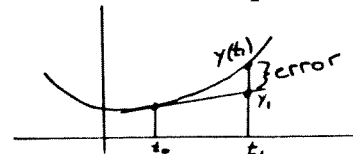
§10. Euler's Method.

The topic of finding approximate solutions of differential equations is a very old one; Euler's method of solving first order equations is probably the oldest.

The idea is very simple. Every first order equation has the form

$$\frac{dy}{dt} = f(t, y),$$

so there is no problem finding the derivative of the solution function; it's given to you. A boundary condition then specifies one value of the solution, namely $y(t_0) = y_0$. So, if you want the solution at any later time t_1 , you can move along the tangent line (the derivative is its slope) to find an approximate value for $y(t_1)$. Pictorially:



Of course, unless the solution curve is always a straight line, you will be wrong in your answer by an error term. Still it is easy to find an estimate for $y(t_1)$ by this method. Call the value you want y_1 . Then, since you have a straight line, you know its equation is

$$(y_1 - y_0) = \left(\frac{dy}{dt} \right)_{t_0} (t_1 - t_0).$$

So

$$y_1 = y_0 + \left(\frac{dy}{dt} \right)_{t_0} (t_1 - t_0), \text{ or}$$

$$(5.1) \quad y_1 = y_0 + f(t_0, y_0) (t_1 - t_0).$$

This last is Euler's formula. Since t_0 , t_1 , y_0 and f are given,

it is easy to find y_1 .

Example 5.1. Suppose

$$y' = f(t, y) = 2yt,$$

where $y(1) = 1$,

and we want $y(1.5)$.

Then $t_0 = 1$, $t_1 = 1.5$, $y_0 = 1$, $\frac{dy}{dt} = 2yt$.

So $(\frac{dy}{dt})_{t_0} = 2y_0t_0 = 2$.

Thus $y_1 = 1 + (0.5)2 = 2 \approx y(1.5)$

This approximation isn't great, since separation of variables says that

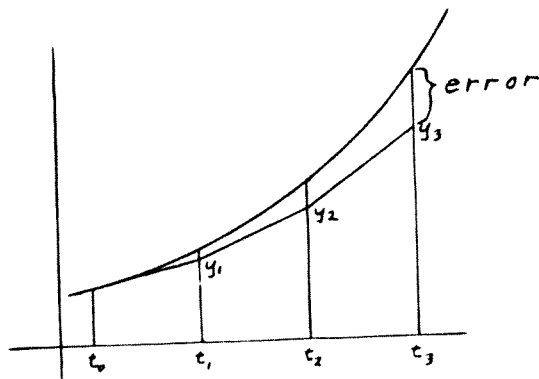
$$y(t) = e^{t^2-1}$$

is the solution of $y' = 2yt$ for $y(1) = 1$; thus the actual value of $y(1.5) = e^{1.25} \approx 3.4904$ (to four decimals).

But the approximation is quick, and can be improved upon.

One such improvement is to increase the number of intervals.

Here is a graph showing a three-interval subdivision:



Suppose you want to go from $t_0 = 1$ to $t_5 = 1.5$ in five steps. You want y_5 , where in each case Euler's formula is used as a recursion formula:

$$y_{i+1} = y_i + (\frac{dy}{dt})_{t_i} (t_{i+1} - t_i), \quad 0 \leq i \leq 4.$$

Since $t_0 = 1$, $t_5 = 1.5$, and there are five intervals, it makes sense to choose each interval of length $\frac{1}{10}$. Thus $t_{i+1} - t_i = 0.1$ for all $0 \leq i \leq 4$. Note that you need not choose the intervals of equal length, but it does make computation slightly easier. When equal intervals are chosen, each $t_{i+1} - t_i$ is the same number, and it is customary to call this number h .

So, in this example

$$h = 0.1 = t_{i+1} - t_i,$$

$$\text{and } y_{i+1} = y_i + (\frac{dy}{dt})_{t_i} \cdot h$$

$$= y_i + f(t_i, y_i) \cdot h, \quad 0 \leq i \leq 4.$$

Now $(\frac{dy}{dt})_{t_0} = 2$ gives $y(1.1) \approx y_1 = y_0 + 2 \cdot (0.1) = 1.2$.

Continuing,

$$y(1.2) \approx y_2 = y_1 + f(t_1, y_1)(0.1)$$

$$= 1.2 + 2(1.1)(1.2)(0.1)$$

$$= 1.464$$

$$y_3 = 1.464 + 2(1.2)(1.464)(0.1)$$

$$= 1.81536$$

$$y_4 = 1.81536 + 2(1.3)(1.81536)(0.1)$$

$$= 2.2873536$$

$$y(1.5) \approx y_5 = 2.2873536 + 2(1.4)(2.2873536)(0.1)$$

$$= 2.927812608$$

Thus increasing the number of intervals has increased the accuracy

of your solution while, at the same time, increasing the amount of computation you or a computer must do. If the computer must do the calculating, you will not mind increasing the number of intervals, otherwise, your feelings on the subject are obvious.

To keep data straight and to correct errors more quickly, you will find a table of data helpful. Below is such a table for the problem we just did.

i	t_i	y_i
0	1	1
1	1.1	1.2
2	1.2	1.464
3	1.3	1.81536
4	1.4	2.2873536
5	1.5	2.927812608

Exercises.

5.1. Find an approximate value for $y(1.5)$ in the example above if the number of intervals is four, each having equal length. How should this answer compare with the one we found using five intervals?

5.2. Consider the differential equation

$$\frac{dH}{dt} = H - 20, \quad H(1) = 25.$$

- Using five intervals of equal length, find an approximate value for $H(3)$.
- Solve the given differential equation by separating variables, and find the exact value of $H(3)$.

§11. Three-Term Taylor Series.

Euler's method of the previous section gives a rather poor approximation for solutions to differential equations unless enormous numbers of intervals are used. However, the method does suggest some more precise techniques. You see, Euler's method is a linearization of a not-necessarily straight-line function. If you look at a Taylor series for a function,

$$f(t) = f(t_0) + f'(t_0)(t-t_0) + \frac{f''(t_0)(t-t_0)^2}{2!} + \dots,$$

the first two terms of the Taylor series are also a linearization of $f(t)$, that is,

$$f(t) \approx f(t_0) + f'(t_0)(t-t_0) = f(t_0) + f'(t_0)h.$$

Thus Euler's method essentially uses the first two terms of Taylor's expansion of the solution function of a differential equation. But why stop after two terms when three will surely give a better approximation? For the three-term Taylor method your recursion formula should be

$$(6.1) \quad y_{i+1} = y_i + f'(t_i, y_i)h + \frac{f''(t_i, y_i)h^2}{2}$$

The only new information you need is f'' . But, since you know $f' = f'(t, y) = y'$, you can find f'' from that.

Example 6.1. Return again to

$$y' = 2yt, \quad y(1) = 1.$$

The question is again to approximate $y(1.5)$, where five intervals

will be used. Thus $h = 0.1$. Next we need to find y'' ; by the product and chain rules

$$y'' = 2ty' + 2y.$$

The recursion formula (6.1) becomes

$$y_{i+1} = y_i + 2t_i y_i (0.1) + \frac{(2t_i (2t_i y_i) + 2y_i)(0.1)^2}{2}.$$

So $y_{i+1} = y_i + 0.2 t_i y_i + 0.01 (2t_i^2 y_i + y_i)$, or

$y_{i+1} = y_i + (0.1)y_i' + \frac{(0.01)}{2} y_i''$. Use of a calculator and a table like the one below helps. Data have been rounded off to four decimals.

i	t_i	y_i	y_i'	y_i''
0	1.0	1.0	2	6
1	1.1	1.23	2.706	8.4132
2	1.2	1.5427	3.7025	11.9714
3	1.3	1.9728	—	—
4	1.4	—	—	—
5	1.5	3.4188		

We have stopped the table above at y_3 . You should continue the computation yourself until you get y_5 , the final

answer, which we have provided.

Having now seen the three-term Taylor method and the Euler method, you can use the fact that both of these are approximations of power series to get an idea of how accurate they are.

Suppose, as our initial condition, we are given $y(a)$, and we are using Euler to approximate $y(b)$, using subintervals of length h . I.e., $a = t_0$, $t_1 = t_0 + h, \dots, t_n = b$. (Of course, $h = \frac{b-a}{n}$). At each step, we are using the linear part of the Taylor series. The error in each of these computations is

$$E_i = \frac{y''(\xi_i)}{2} h^2,$$

where ξ_i is chosen in the i -th subinterval. Since the computation is iterated n times, our total error is bounded by

$$\left| \frac{y''(\xi)}{2} h^2 \right| \cdot n = \left| \frac{y''(\xi)}{2} (b-a) \right| h$$

Here ξ is chosen in $[a, b]$ to maximize y'' , and we are using the fact that $n = \frac{b-a}{h}$.

Since our error is bounded by an expression linear in h , we say the error is of order h , where h is the length of the subintervals used. By analogous reasoning, we find that the error in the three-term Taylor method is of order h^2 . So, given a calculator or computer, use Taylor.

Exercises

6.1 Finish the table in the example above by finding y_3' , y_3'' , y_4 , y_4' , y_4'' and y_5 . Verify that, to eight decimals, y_5 is the correct approximation by the three-term Taylor formula.

6.2 Use the three-term Taylor method to find a three decimal approximation to the indicated value if $h = 0.1$.

- $y' = 2x - y + 1$, $y(1) = 2$; find $y(1.5)$.
- $y' = e^{-y}$, $y(0) = 0$; find $y(0.5)$.

§12. The Runge-Kutta Method.

In the last section we saw that the Euler and three-term Taylor methods have errors on the orders of h and h^2 , respectively, where h is the length of the interval in the approximation. This means that if we double the number of intervals (and therefore halve each length), Euler's method will have about half as much error as the original approximation, while Taylor's method will have one-fourth the error. In this section we will consider a method of approximation of solutions of differential equations whose error is on the order of h^4 , making it highly accurate, since doubling the number of intervals used in this method decreases the error by a factor of 16. In return, the method has a large disadvantage for hand computation: it requires using values at four points at each stage of computation.

Consider again $y' = f(t, y)$. The technique we show is called the Runge-Kutta method (some books call it the fourth-order Runge-Kutta method) and is based on finding appropriate constants a, b, c, d so that

$$y_{i+1} = y_i + ak_1 + bk_2 + ck_3 + dk_4,$$

where

$$\begin{aligned} k_1 &= hf(t_1, y_1) \\ k_2 &= hf(t_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1) \\ k_3 &= hf(t_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2) \\ k_4 &= hf(t_1 + h, y_1 + k_3). \end{aligned}$$

The constants used in the method are

$$a = \frac{1}{6}, \quad b = \frac{1}{3}, \quad c = \frac{1}{3}, \quad d = \frac{1}{6}.$$

This gives the recursion formula

$$(7.1) \quad y_{i+1} = y_i + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4].$$

Without giving a derivation of the Runge-Kutta formula, we note that the expression in formula (7.1) above depends on k_1, k_2, k_3, k_4 , and is a weighted average of these numbers (the weights being $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$). Since each of k_1, k_2, k_3, k_4 is a function of

$$y' = f(t, y),$$

the function in the brackets is then a weighted average of the slope evaluated at different points. Those of you who have seen Simpson's rule for approximating integrals in a calculus course have already seen a reduced form of the Runge-Kutta method; if $f(t, y) = f(t)$, then $k_2 = k_3$ and

$$y_{i+1} = y_i + \frac{h}{6} [f(t_1) + 4f(t_1 + \frac{h}{2}) + f(t_1 + h)],$$

which is Simpson's rule.

Example 7.1. Use Runge-Kutta on the example of the previous sections:

$$y' = 2ty, \quad y(1) = 1.$$

Find $y(1.5)$, and compare the answer with the approximate solutions by Euler and Taylor, as well as with the exact solution given by integration.

Solution: First we will compute k_1 through k_4 when $i = 0$. We know that $h = 0.1$, $t_0 = 1$, $y_0 = 1$ as before.

$$\begin{aligned} \text{So} \quad k_1 &= hf(t_0, y_0) = (0.1) \cdot 2 \cdot 1 \cdot 1 = 0.2, \\ k_2 &= hf(t_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) \\ &= h \cdot 2(t_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) \\ &= (0.1) \cdot 2(1.05) (1.1) = 0.231 \\ k_3 &= hf(t_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) \\ &= (0.1) \cdot 2(1.05) (1.1055) \\ &= 0.234255 \end{aligned}$$

$$\begin{aligned} k_4 &= hf(t_0 + h, y_0 + k_3) \\ &= (0.1) \cdot 2(1.1) (1.234255) \\ &= 0.2715361 \end{aligned}$$

Now we can find y_1 by (7.1) ,

$$\begin{aligned} y_1 &= y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \\ &= 1 + \frac{1}{6} (0.2 + 0.462 + 0.46851 + 0.2715361) \\ &= 1.23367435 \\ &= 1.2337 \text{ (to four decimals).} \end{aligned}$$

Given y_1 , we now must calculate new k_1, k_2, k_3, k_4 to find y_2 ,

then continue through y_5 . The table follows:

1	t_1	y_1
0	1.0	1.0
1	1.1	1.2337
2	1.2	1.5527
3	1.3	1.9937
4	1.4	2.6116
5	1.5	3.4902

Back in section 5 we found the actual answer to be $y(1.5) = 3.4904$. Thus our approximation by the Runge-Kutta method is accurate through the first three decimal places. Further, this compares with $y_5 = 2.9278$ by Euler, and $y_5 = 3.4188$ by three-term Taylor methods. Clearly, if we are given a calculator, or better yet, a computer, Runge-Kutta must be the preferred method. So learn it well!

Exercises

1.1. Use Runge-Kutta to find a three-decimal approximation to the indicated value if $h = 0.1$.

a) $y' = 2x - y + 1$, $y(1) = 2$; find $y(1.5)$.

b) $y' = x + y^2$, $y(0) = 1$.

1.5. Use the formula for Simpson's rule given in this section to approximate $\int_0^6 \sqrt{4+x^3} dx$.

SOLUTIONS TO EXERCISES

Exercise 2

The separable equations are:

#1. $y \ln y \, dx + (1+x^2)dy = 0$

$$\int \frac{dx}{1+x^2} = - \int \frac{dy}{y \ln y}$$

$$\arctan x + c = - \ln(\ln y)$$

#2. $e^{-y} dy = e^{-x} dx$

$$\int e^{-y} dy = \int e^{-x} dx$$

$$e^{-y} = e^{-x} + C$$

$$\ln(e^{-y}) = \ln(e^{-x} + C)$$

$$y = -\ln(C + e^{-x})$$

#3, #4 cannot be separated.

#5. $\sqrt{y} \, dy = \frac{dx}{\sqrt{2x}}$

$$\int \sqrt{y} \, dy = \int \frac{dx}{\sqrt{2x}}$$

$$\frac{2}{3} y^{3/2} = \sqrt{2x} + C'$$

$$y = \left(\frac{3}{2} \sqrt{2x} + C \right)^{2/3} \text{ where } C = \frac{3}{2} C'$$

#6, #7, #8 cannot be separated.

#9. $\frac{d^2 y}{dt^2} + \frac{dy}{dt} = 0$

We can separate this equation by making the following substitution:

$$u = \frac{dy}{dt}, \quad \frac{du}{dt} = \frac{d^2 y}{dt^2}$$

$$\frac{du}{dt} + u = 0$$

$$\frac{du}{u} = -dt$$

$$\ln u = -t + C$$

or

$$u = Ce^{-t} \quad (\text{note: This is not the same } C \text{ as above.})$$

Thus we get,

$$\frac{dy}{dt} = Ce^{-t}$$

$$y = -Ce^{-t} + C'$$

C, C' are arbitrary constants

#10, #11, #12 cannot be separated.

#13. $x \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0$

Let $u = \frac{dy}{dx}$, $\frac{du}{dx} = \frac{d^2 y}{dx^2}$ the eqn. becomes,

$$x \frac{du}{dx} + u = 0$$

$$\frac{du}{u} = - \frac{dx}{x}$$

Solution is: $u = - \frac{C_1}{x}$ or $\frac{dy}{dx} = - \frac{C_1}{x}$

$$y = -C_1 \ln x + C_2$$

Exercise 3.

#1, #2 are non-homogeneous

#3. $(x^2 + y^2)dy - y^2 dx = 0$ which reduces to:

$$\frac{dy}{dx} = \frac{(y/x)^2}{1 + (y/x)^2}$$

Let $v = y/x$, $y = vx$, $\frac{dy}{dx} = v + x \frac{dv}{dx}$ with this substitution we get

$$v + x \frac{dv}{dx} = \frac{v^2}{1+v^2}$$

$$\int \frac{dx}{x} = - \int \frac{1+v^2}{v^2-v+1} \frac{1}{v} dv$$

$$\ln x = \ln v + \frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}\left(v - \frac{1}{2}\right)\right) + C$$

or,

$$\ln x = -\ln(y/x) - \frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}\left(y/x - \frac{1}{2}\right)\right) + C$$

#4, #5 are non-homogeneous.

#6. $(xe^{y/x} + y)dx - x dy = 0$

$$\frac{dy}{dx} = e^{y/x} + y/x, \text{ let } v = y/x$$

$$x \frac{dv}{dx} + v = e^v + v$$

$$\int e^{-v} dv = \int \frac{dx}{x}, e^{-v} = -\ln x + C$$

$$v = -\ln(C - \ln x)$$

$$y = -x \ln(C - \ln x)$$

#7, #8, #9, #10 are non-homogeneous.

#11. $(x-2y)dy + y dx = 0$

$$\frac{dy}{dx} = \frac{-y/x}{1-2(y/x)}$$

$$v = y/x$$

$$x \frac{dv}{dx} + v = \frac{-v}{1-2v}$$

$$\int \frac{1-2v}{v^2-v} dv = 2 \int \frac{dx}{x}$$

$$-\ln(v^2-v) = 2 \ln x + C = \ln x^2 + C$$

$$\left(\frac{y}{x}\right)^2 - \frac{y}{x} = \frac{C}{x^2} \quad C \text{ is an arbitrary constant}$$

#13 is non-homogeneous".

Exercise 4. The first order linear equations are:

#4. $\frac{dy}{dx} + 2y = e^{-x}$ multiply by integrating factor $e^{\int 2dx} = e^{2x}$ we'll get

$$e^{2x} y = \int e^{2x} e^{-x} dx + C = e^x + C$$

$$y = e^{-x} + Ce^{-2x}$$

#7. $\frac{dy}{dx} + \frac{1}{x}y = \frac{\sin x}{x}$ Integrating factor is $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$
multiply the equation by x and we'll obtain

$$xy = \int \sin x dx = -\cos x + C$$

$$y = -\frac{\cos x}{x} + \frac{C}{x}$$

Exercise 5

The exact differential equations are:

#8. $(x+y)dx + (x+y^2)dy = 0$

$$M = x+y \quad \frac{\partial M}{\partial y} = 1$$

$$N = x+y^2 \quad \frac{\partial N}{\partial x} = 1$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact.

$$M = \frac{\partial f}{\partial x} = x+y, \quad f(x,y) = \int (x+y)dx + C(y)$$

$$f(x,y) = \frac{x^2}{2} + yx + C(y)$$

we need to evaluate $C(y)$ $N = \frac{\partial f}{\partial y}(x,y)$ So we get

$$x+y^2 = x + c'(y)$$

$$c'(y) = y^2, \quad c(y) = y^3/3 + C$$

$$\text{So } f(x,y) = \frac{x^2}{2} + yx + y^3/3 + C$$

Since $df = 0$ we conclude $f(x,y) = \text{constant}$. Combining constants we get

$$\frac{x^2}{2} + yx + \frac{y^3}{3} = \text{constant}$$

#10. $(2xe^y + e^x)dx + (x^2+1)e^y dy = 0$

$$M = 2xe^y + e^x \quad N = (x^2+1)e^y$$

$$\frac{\partial M}{\partial y} = 2xe^y \quad \frac{\partial N}{\partial x} = 2xe^y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{Exact}$$

$$\frac{\partial f}{\partial x}(x,y) = M = 2xe^y + e^x, \quad f(x,y) = \int (2xe^y + e^x)dx + c(y)$$

$$f(x,y) = x^2e^y + e^x + c(y)$$

$$N = \frac{\partial f}{\partial y} = (x^2+1)e^y = x^2e^y + c'(y)$$

$$c'(y) = e^y, \quad c(y) = e^y + C$$

$$f(x,y) = \text{constant} = x^2e^y + e^x + e^y + C \quad \text{combining constants we'll get}$$

$$(x^2+1)e^y + e^x = C$$

Solving for y , the equation becomes,

$$y = \ln\left(\frac{C-e^x}{x^2+1}\right)$$

#11. $(x-2y)dy + ydx = 0$

$$M = y \quad N = x-2y$$

$$\frac{\partial M}{\partial y} = 1 \quad \frac{\partial N}{\partial x} = 1 \quad \therefore \text{Exact}$$

$$\frac{\partial f}{\partial x} = (x,y) = M, \quad f(x,y) = \int ydx = xy + c(y)$$

$$\frac{\partial f}{\partial y} = N \quad x - 2y = x + c'(y)$$

$$c(y) = -y^2 + C$$

$$\text{Const} = yx - y^2 + C$$

$$\text{or } C = yx - y^2$$

Exercise #6

The second order equations are:

#9. $\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$ Let $p = \frac{dy}{dt}, \quad \frac{dp}{dt} = \frac{d^2y}{dt^2}$

after substitution the equation becomes,

$$\frac{dp}{dt} + p = 0$$

$$p(t) = c_1 e^{-t}$$

also $p = \frac{dy}{dt} = c_1 e^{-t}$ integrating, we get

$$y(t) = -c_1 e^{-t} + c_2$$

Since $-c_1$ is arbitrary we write $-c_1$ simply as c_1

$$y(t) = c_1 e^{-t} + c_2$$

#12. $\frac{d^2 y}{dx^2} + w^2 y = 0 \quad w \neq 0$

Let $p = \frac{dy}{dx}$, then $\frac{d^2 y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \frac{dy}{dx} = p \frac{dp}{dy}$

Then the equation becomes,

$$p \frac{dp}{dy} + w^2 y = 0$$

This equation is separable

$$p = \sqrt{c - w^2 y^2} \quad c \text{ is arbitrary}$$

$$p = \frac{dy}{dx} = \sqrt{c - w^2 y^2}$$

$$\int \frac{dy}{\sqrt{c - w^2 y^2}} = \int dx$$

$$\frac{1}{w} \sin^{-1}\left(\frac{w}{\sqrt{c}} y\right) = x + c'$$

$$y(x) = \frac{\sqrt{c}}{w} \sin(w(x+c'))$$

c, c' are arbitrary constants

#13. $x \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0, \quad p = \frac{dy}{dx}, \quad \frac{dp}{dx} = \frac{d^2 y}{dx^2}$

$$x \frac{dp}{dx} + p = 0$$

$$p = c/x$$

$$\frac{dy}{dx} = \frac{c}{x}$$

$$y = c \ln x + c'$$

Exercise #7

#1. $\frac{d^2 y}{dt^2} - y = 0$

The corresponding algebraic equation is :

$$r^2 - 1 = 0 \quad (r+1)(r-1) = 0$$

The solution is:

$$y = Ae^t + Be^{-t}$$

#2. $\frac{d^2 y}{dt^2} + y = 0$

$$r^2 + 1 = 0 \quad \text{the roots are imaginary, } \pm i.$$

So the solution is :

$$y = A'e^{it} + B'e^{-it}$$

If we want the solution to be put in terms of real functions instead of imaginary ones we can achieve this by using Euler's equation.

$$e^{it} = \cos t + i \sin t$$

$$e^{-it} = \cos t - i \sin t$$

$$y = A'e^{it} + B'e^{-it} = (A^1 + B^1) \cos t + (A^1 - B^1) i \sin t$$

But A^1, B^1 are arbitrary, so we can write the above equation simply as:

$$y = A \cos t + B \sin t$$

(A, B are arbitrary and may be complex)

#3. $y'' + 6y' + 5y = 0 \quad r^2 + 6r + 5 = 0$

roots are -5, -1.

Solution is:

$$y = Ae^{-5t} + Be^{-t}$$

*4. $y'' - 10y' + 16y = 0$ $r^2 - 10r + 16 = 0$

roots are 8, 2

$$y = Ae^{8t} + Be^{2t}$$

*5. $y^{(4)} - 4y'' + 4y = 0$

$$r^4 - 4r^2 + 4 = 0$$

$$(r - \sqrt{2})^2 (r + \sqrt{2})^2 = 0$$

$$r = +\sqrt{2} \text{ Double root}$$

$$r = -\sqrt{2} \text{ Double root}$$

The solution is:

$$y = Ae^{\sqrt{2}x} + Bxe^{\sqrt{2}x} + Ce^{-\sqrt{2}x} + Dxe^{-\sqrt{2}x}$$

*6. $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

$$(r-1)^2(r+2) = 0$$

$$r = 1 \text{ double root}$$

$$r = -2$$

$$y = Ae^{-2x} + Be^x + Cx e^x$$

*7. $(D-6)^4(D+2-i)(D+2+i)(D-\sqrt{2})Dy = 0$

$$(r-6)^4(r-(i-2))(r-(-2-i))(r-\sqrt{2})r = 0$$

roots are

$$r = 6 \quad 4 \text{ roots}$$

$$r = -2+i \quad \text{complex conjugates}$$

$$r = -2-i$$

$$r = \sqrt{2}$$

$$r = 0$$

Before we write down the generalization, let's first express the solution corresponding to the complex roots in real form by using Euler's equation:

that is, $Ae^{(-2+i)x} + Be^{(-2-i)x}$ is to be expressed in terms of

real functions

$$e^{(-2+i)x} = e^{-2x}e^{ix} = e^{-2x}(\cos x + i \sin x)$$

$$e^{(-2-i)x} = e^{-2x}e^{-ix} = e^{-2x}(\cos x - i \sin x)$$

$$\begin{aligned} \text{thus } Ae^{(-2+i)x} + B'e^{(-2-i)x} &= (A'+B')e^{-2x} \cos x - (A'-B')e^{-2x} \sin x \\ &= Ae^{-2x} \cos x + Be^{-2x} \sin x \end{aligned}$$

Now we can continue to write the entire solution

$$y = Ae^{-2x} \cos x + Be^{-2x} \sin x + Ce^{\sqrt{2}x} + De^{6x} - Exe^{6x} + Fx^2 e^{6x} - Gx^3 e^{6x} + H$$

*9. $e^{iBx} = \cos Bx + i \sin Bx$

then

$$e^{-iBx} = e^{i(-B)x} = \cos(-Bx) + i \sin(-Bx)$$

But $\cos(-Bx) = \cos Bx$

$$\sin(-Bx) = -\sin Bx$$

So

$$e^{-iBx} = \cos Bx - i \sin Bx$$

Exercise 5

#2a. $y'' + 2y' - 3y = 6x + 2$

First, look for the solution of the homogeneous equation

$$y'' + 2y' - 3y = 0 \quad r^2 + 2r - 3 = 0 \quad (r+3)(r-1) = 0$$

Solution is

$$y_h = Ae^{-3x} + Be^{+x}$$

look for a particular solution of the form

$$y_p = cx + D$$

$$y'_p = c$$

$$y''_p = 0$$

This solution must satisfy $y''_p + 2y'_p - 3y_p = 6x + 2$

So we must have

$$2c - 3(cx + D) = 6x + 2$$

equating coefficients $-3c = 6$, $c = -2$

$$2c - 3D = 2, D = -2$$

So the entire solution is

$$y = y_h + y_p = Ae^{-3x} + Be^{+x} - 2x - 2$$

Note: We don't multiply the solution y_p by an arbitrary constant

#2b. $y'' - y = e^{2x}$

The associated algebraic eqn is: $r^2 - 1 = 0$ $(r+1)(r-1)$

roots $r = \pm 1$

$$\text{solution } y_h = Ae^x + Be^{-x}$$

Assume a particular solution is of the form,

$$y_p = ce^{2x}$$

$$y''_p = 4ce^{2x}$$

$$e^{2x} = y''_p - y_p = 4ce^{2x} - ce^{2x}$$

$$\text{so } 3c = 1, c = \frac{1}{3}$$

$$y_p = \frac{1}{3} e^{2x}$$

General Solution is:

$$y = y_h + y_p = Ae^x + Be^{-x} + \frac{1}{3} e^{2x}$$

#3. $y'' - y = e^x$

$$\text{from above } y_h = Ae^x + Be^{-x}$$

We cannot use e^x as our particular since it is a solution to the homogeneous equation. So, try

$$y_p = cxe^x$$

$$y'_p = cxe^x + ce^x$$

$$y''_p = cxe^x + 2ce^x$$

Solving for c ,

$$cxe^x + 2ce^x - cxe^x = e^x$$

$$2c = 1, c = \frac{1}{2}, y_p = \frac{1}{2} xe^x$$

general solution is!

$$y = Ae^x + Be^{-x} + \frac{1}{2} xe^x$$

HARDER QUESTIONS

Question 2

(E), $Mdx + Ndy = 0$

$$\frac{dy}{dx} = -\frac{M}{N}$$

The solutions which correspond to curves perpendicular to (E) then satisfy

$$\frac{dy}{dx} = \frac{N}{M}$$

or $xy - xdx = 0$

Now let's find the solution to

$$2xydy + (x^2 - y^2)dx = 0$$

divide by xy

$$2dy + \left(\frac{x}{y} - \frac{y}{x}\right)dx = 0$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{y}{x} - \frac{x}{y}\right)$$

Use the substitution $v = y/x$ $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$$v + x \frac{dv}{dx} = \frac{1}{2} \left(v - \frac{1}{v}\right) \quad x \frac{dv}{dx} = -\left(\frac{v^2+1}{2v}\right)$$

$$\frac{2v}{v^2+1} dv = -\frac{dx}{x}$$

$$\ln(v^2+1) = -\ln x + c'$$

$$v^2 = \frac{c}{x} - 1$$

$$(y/x)^2 = \frac{c}{x} - 1$$

$$y = \pm x \sqrt{\left(\frac{c}{x} - 1\right)}$$

The perpendicular solution satisfy:

$$(y^2 - x^2)dy + 2xydx = 0$$

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2} = \frac{2}{\frac{x}{y} - \frac{y}{x}}$$

Let $v = y/x$

$$v + x \frac{dv}{dx} = \frac{2}{\frac{1}{v} - v} = \frac{2v}{1-v^2}$$

$$\text{so } x \frac{dv}{dx} = \frac{2v}{1-v^2} - \frac{v(1-v^2)}{1-v^2} = \frac{v+v^3}{1-v^2}$$

$$\int \frac{1}{v} \frac{1-v^2}{1+v^2} dv = \int \frac{dx}{x}$$

The integral on the left is solved by partial fractions.

$$\frac{1}{v} \frac{1-v^2}{1+v^2} = \frac{1}{v} - \frac{2v}{v^2+1}$$

$$\int \frac{1-v^2}{v(1+v^2)} dv = \ln v - \ln|v^2+1| = \ln \frac{v}{v^2+1}$$

$$\text{So } \ln \frac{v}{v^2+1} = \ln x + c'$$

$$\frac{v}{v^2+1} = cx$$

$$\text{or } y = c(y^2+x^2)$$

Question #4

$$F = ma = m \frac{d^2x}{dt^2} = -kx \quad \text{by Hooke's Law.}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$r^2 + \frac{k}{m} = 0$$

$$r = \pm i \sqrt{\frac{k}{m}}$$

So the real solution is

$$x = c_1 \sin \sqrt{\frac{k}{m}} t + c_2 \cos \sqrt{\frac{k}{m}} t$$

We are given that the body is released from rest ($\dot{x}(0) = 0$) at a distance

A from equilibrium ($x(0) = A$). Using this data, we determine c_1

and c_2 :

$$\dot{x} = c_1 \sqrt{\frac{k}{m}} \cos \sqrt{\frac{k}{m}} t - c_2 \sqrt{\frac{k}{m}} \sin \sqrt{\frac{k}{m}} t$$

$$\text{so } 0 = \dot{x}(0) = c_1 \sqrt{\frac{k}{m}}; \text{ thus } c_1 = 0.$$

$$A = x(0) = c_2 \quad \text{so our solution is:}$$

$$x(t) = A \cos \sqrt{\frac{k}{m}} t$$

Question #5

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = \frac{dE}{dt}$$

$$\text{a. } R = 0, \quad \frac{1}{LC} = \omega^2 \quad E = \text{constant}$$

divide by L.

$$\frac{d^2 I}{dt^2} + \omega^2 I = 0$$

The solution is:

$$I = c_1 \cos \omega t + c_2 \sin \omega t$$

b. $R = 0$, $\frac{1}{LC} = \omega^2$, $E = A \sin \alpha t$ $\alpha \neq \omega$

$$\frac{d^2 I}{dt^2} + \omega^2 I = \frac{\alpha A}{L} \cos \alpha t$$

Solution to the homogeneous equation is:

$$I_h = c_1 \cos \omega t + c_2 \sin \omega t$$

The particular solution is of the form

$$I_p = c_3 \cos \alpha t + c_4 \sin \alpha t, \text{ need to find } c_3, c_4$$

$$\frac{d^2 I_p}{dt^2} = -\alpha^2 c_3 \cos \alpha t - \alpha^2 c_4 \sin \alpha t$$

Inserting this into the equation we get

$$c_3(-\alpha^2 + \omega^2) \cos \alpha t + c_4(-\alpha^2 + \omega^2) \sin \alpha t = \frac{\alpha A}{L} \cos \alpha t$$

$$\text{hence } c_4 = 0$$

$$c_3 = \frac{\alpha A}{L} \frac{1}{\omega^2 - \alpha^2}$$

So the solution is

$$I = I_h + I_p = c_1 \cos \omega t + c_2 \sin \omega t + \frac{\alpha A}{L} \frac{\cos \alpha t}{(\omega^2 - \alpha^2)}$$

(c) $R = 0$, $\frac{1}{LC} = \omega^2$, $E = A \sin \omega t$

$$\frac{d^2 I}{dt^2} + \omega^2 I = \frac{\omega A}{L} \cos \omega t$$

$$I_h = c_1 \cos \omega t + c_2 \sin \omega t$$

assume a particular solution of the form.

$$I_p = C_3 t \cos \omega t + C_4 \sin \omega t$$

$$I_p'' = -\omega C_3 \sin \omega t - \omega^2 C_3 t \cos \omega t - \omega C_3 \sin \omega t + \omega C_4 \cos \omega t - \omega^2 C_4 \sin \omega t + \omega C_4 \cos \omega t.$$

$$I_p'' + \omega^2 I_p = -2\omega C_3 \sin \omega t + 2\omega C_4 \cos \omega t = \frac{\omega A}{L} \cos \omega t$$

it follows that $C_3 = 0$ and $C_4 = \frac{A}{2L}$

So $I_p = \frac{A}{2L} t \sin \omega t$

$$I = I_h + I_p = c_1 \cos \omega t + c_2 \sin \omega t + \frac{A}{2L} t \sin \omega t$$