Mathematics Support Capsules

GRAPHING

This module includes the following capsules:

These Mathematics Support Capsules were prepared under the supervision of Beverly West, Lecturer, Department of Mathematics, Cornell University, Ithaca, New York 14853, with assistance from King Chan, Ken Gardner, Kristen Jackson, Ann Michel, Ken Okamoto, and funding from the Exxon Corporation. The authors solicit your comments, corrections, and suggestions for future revision.

August, 1981

Mathematics Support Capsules

GRAPHING

0. DIAGNOSTIC TEST

1) Plot points, and draw the graph for

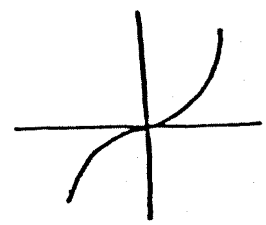
$$x^2 = 3 - y$$

2) Which of the following are symmetrical about the y-axis? ____________

Which are symmetrical about the origin? ____________

a) $y = x$ b) $y = x^3$ c) $y = x^2$ d) $y = 2x + 3$ e) $y = |x|$

3) What equation do you expect for the following graph?



4) Sketch a graph of $f(x) = -(x+2)^2$ (without plotting more than one or two points).

5) Write an equation for the line containing the points: (0,1) (4,2).

6) Sketch: $y = (x+1)^3(x+2)^2x^4$ (There is a neat way to do this without multiplying out or plotting zillions of points. See how far you can get in a couple of minutes, and then go on.)

7) Find horizontal and vertical asymptotes for:

a) $y = \frac{1}{x+1}$ b) $y + 2 = \frac{1}{(x+1)(x+2)}$

8) Graph and identify $(x-2)^2 + y^2 = 16$

9) Graph and identify $\frac{1}{x+1}$

For solutions and referrals, see next page:

*References

Deborah Hughes-Hallett, <u>The Math Workshop: Elementary Functions</u> (W.W.Norton, 1980)

Available in Mathematics Support Center and at local bookstores.

V. Frederick Rickey, "Qualitative Graphing Techniques"

On reserve in Mathematics Library and Mathematics Support Center.

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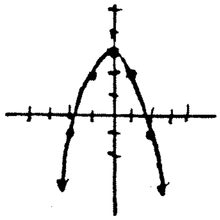
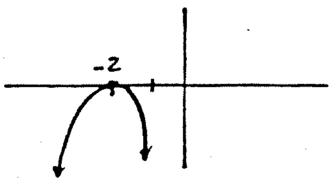
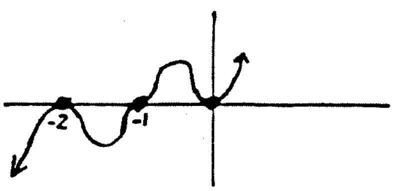
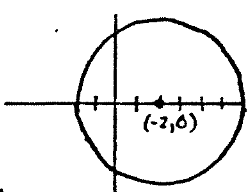
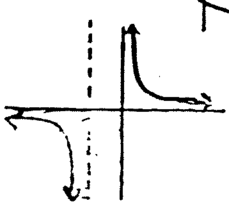
	Answers to Diagnostic Test	If you missed these questions, review the indicated Graphing capsules	If you need more than a brief review, work through the indicated sections of Hughes-Hallett *Elementary Functions**
1)	x | y 0 | 3 $\pm\sqrt{3}$ | 0 ± 1 | 2 ± 2 | -1 	I Basics	
2)	c,e y-axis a,b origin	II Symmetry	Chap.3, sec.2
3)	$y = x^3$,x ,etc.	III Graphs You Should Know	p. 9
4)		IV New Graphs From Old	Chap.3, sec.3,4,5
5)	$y = \frac{1}{4}x + 1$ (or an equivalent form like $\frac{y-2}{x-1} = \frac{1}{4}$)	V Straight Lines	Chap. 2
6)		VI Polynomials	Chap.4, sec.1 *Also,especially,See Rickey*
7)	a) horiz $y = 0$ vert. $x = -1$ b) horiz $y = -2$ vert. $x = -1,-2$	VII Finding Asymptotes	Chap.4, sec.2 See also Rickey*
8)	 circle center (2,0) radius (4)  hyperbola asymptotes at $x=-1, y=0$	VIII Conic Sections	Chap.5 * References on p. O-2

Mathematics Support Capsules

GRAPHING

I. BASICS

Plotting Points:

Point plotting tends to be a lengthy and often cumbersome way to construct graphs, but it will always give you a picture of the relation (eventually), and it is a method you can always fall back on.

For the relations at the right, set up a table for x and y values that satisfy the relation. Find 4, 5 or more points, plot them, and draw a smooth curve(s) connecting them.

Hints:

1) Try easy points: $x = 0$ (y-intercept),
$y = 0$ (x-intercept),
$x = \pm 1, \pm 2, \ldots$

Don't forget negative numbers.
Try $x \to \infty$ and $x \to -\infty$

2) Watch out for zero in the denominator.
$\frac{a}{0}$ is undefined, so there will be no points on the graph where the denominator = 0.

3) Watch out for radicals:

a) $\sqrt{\text{negative number}}$ cannot be graphed on real axes; such numbers are undefined in the real number system.

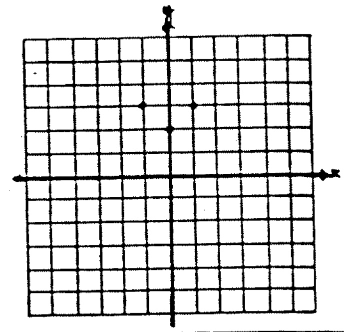
b) the symbol $\sqrt{}$ refers to the positive square root. E.g., $\sqrt{25} = 5$ (although $x^2 = 25$ has two roots, $x = \pm\sqrt{25} = \pm 5$).

4) For $x^2 = 36$, remember that x can be <u>both</u> $+6$ and -6.

5) Don't determine your scale on the axes until you've seen the points to plot.

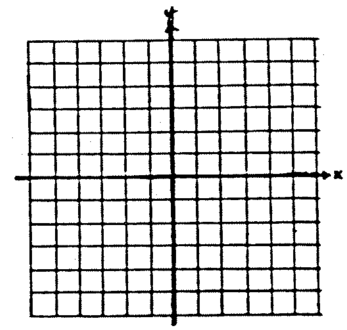
A. $y = x^2 + 2$

x	y
0	2
±1	3
$\to \infty$	$\to \infty$
$\to -\infty$	$\to \infty$



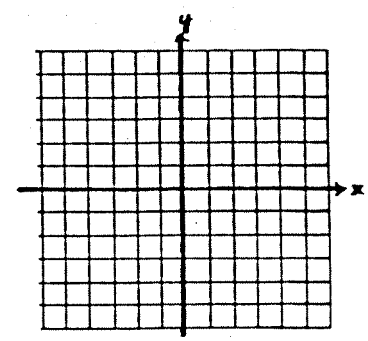
B. $y = 3x + 2$

x	y



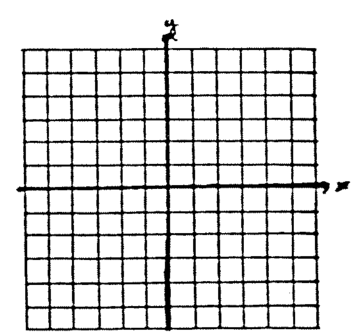
C. $x^2 + y^2 = 9$

x	y



D. $y = \frac{1}{x+3}$

x	y

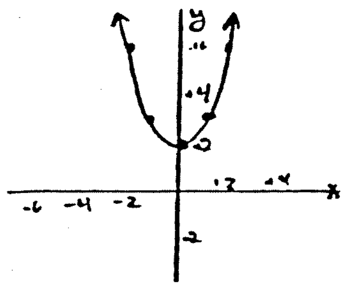


Solutions to graphs from last page:

More about these graphs is contained in subsequent graphing capsules. However these points (or a few similar ones) should be enough to show the character of the graphs of these relations. If your graphs are different or incomplete, look for what crucial information you missed.

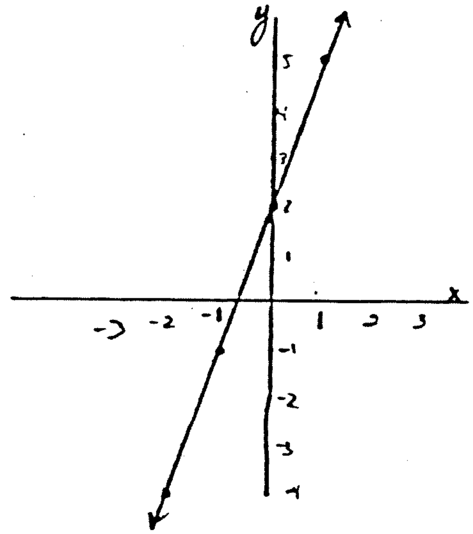
7. $y = x^2 + 2$

x	y
0	2
no good! ← $\sqrt{-2}$ ←	0
± 1	3
± 2	6
$\to \infty$	$\to \infty$
$\to -\infty$	$\to \infty$

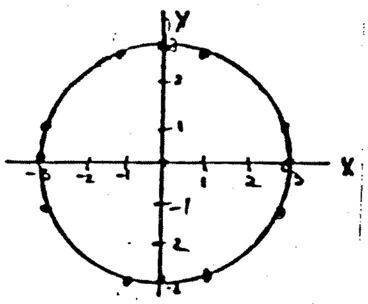


8. $y = 3x + 2$

x	y
-2	-4
-1	-1
0	2
1	5
2	8
$\to \infty$	$\to \infty$
$\to -\infty$	$\to -\infty$



C. $x^2 + y^2 = 9$

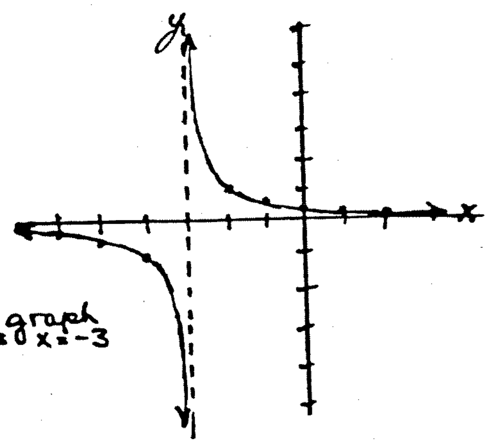


x	y
0	± 3
± 3	0
± 1	$\pm\sqrt{8} = \pm 2\sqrt{2} \approx \pm 2.8$
$\pm\sqrt{8}$	± 1

note: no points possible for $|x| > 3$ or $|y| > 3$.

D. $y = \frac{1}{x+3}$

x	y
0	$1/3$
1	$1/4$
2	$1/5$
-1	$1/2$
-2	1
-3	undefined - graph can't cross $x = -3$
-4	-1
-5	$-1/2$
-6	$-1/3$
$\to \infty$	0^+
$\to -\infty$	0^-



This point plotting is the basis for all graphing. In the capsules that follow, you will be shown a number of shortcuts that will quicken the graphing process. But you will still want to plot <u>some</u> points.

Determining excluded regions:

An important aspect of graphing is deciding where in the plane the graph lies. Many regions can thereby be excluded from future consideration.

To determine excluded regions for the graph of $y = f(x)$ ask the following questions:

$\boxed{\text{What } x \text{ values are permitted?}}$

Exclude values of x that give a zero denominator.

Exclude values of x that give a negative number inside a radical of even index.

$\boxed{\text{What range of } y\text{-values will result?}}$

When is y positive and when is it negative?

Are there any limits on y?

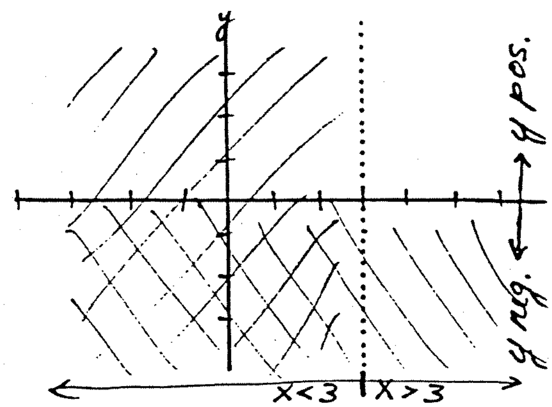
Crosshatch any excluded regions; dash a line for a single value.

Examples:

$y = \sqrt{x-3}$

x: must be ≥ 3 (so that $x-3 \geq 0$)

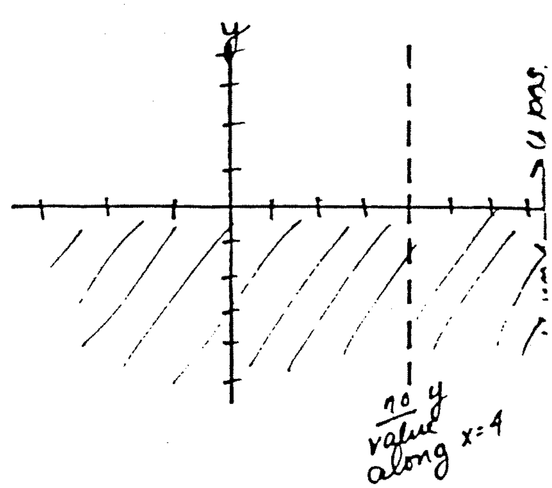
y: must be > 0 (because $\sqrt{}$ means the positive root.)



$y = \dfrac{1}{(x-4)^2}$

x: can be anything except $x = 4$

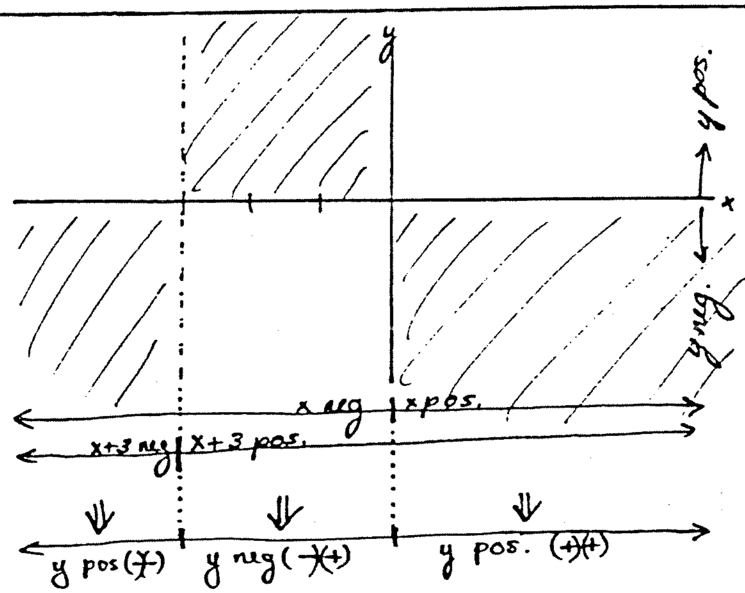
y: always > 0 (because denominator is squared.)



$y = x(x+3)$

x: can be anything

y: sign depends on factors - analysis below graph.



These graphs are not finished, but you know now that they must lie in the regions that are not cross-hatched. Sketch the rough graphs using as few key points as you can. Solutions at bottom of next page.

In graphing polynomials and rational functions (Capsules VI and VII), determining the excluded regions of a graph will save much work.

See if you can determine the excluded regions for the following graphs:

E. $y = \sqrt{7 - 2x}$

F. $y = \dfrac{x^3}{x-1}$

G. $y = \sin x$

Solutions:

E. x: $7 - 2x \geq 0$

$-2x \geq 7$

$x \leq 7/2 = 3.5$

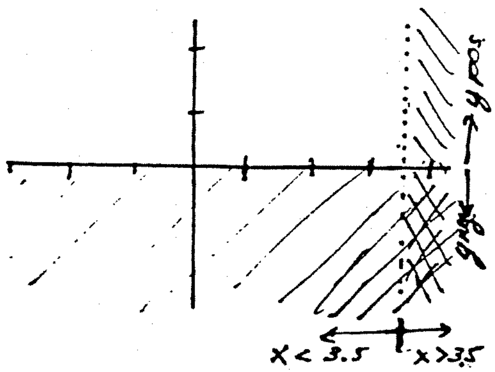
y: $\sqrt{\ }$ requires y pos.

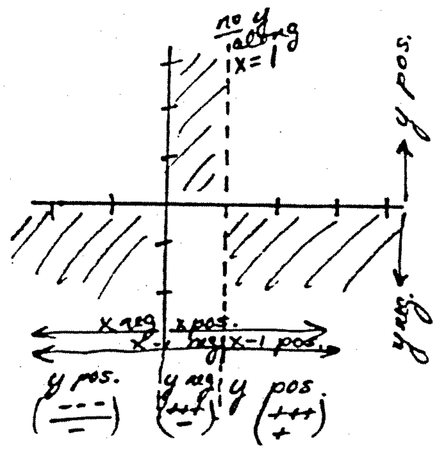
F. x: $\neq 1$

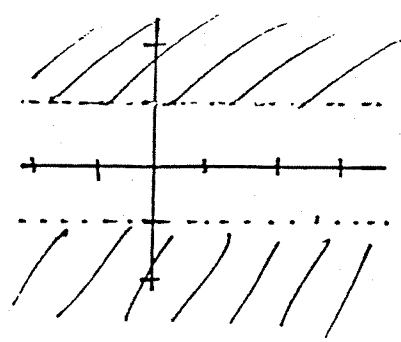
y: pos. & neg. shown by factors below.

G. x: all real numbers

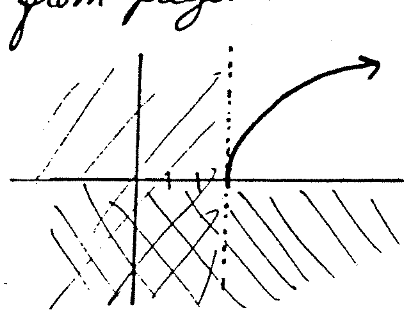
y: $|\sin x| \leq 1$



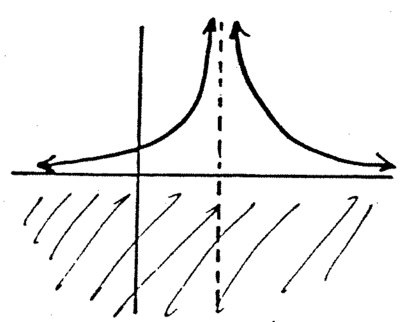




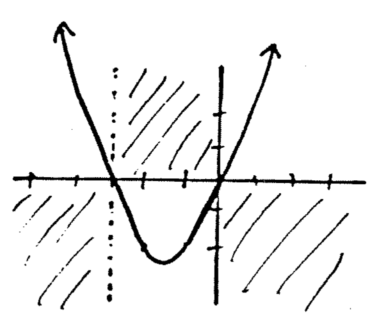
from pages 3 and 4



$y = \sqrt{x-3}$



$y = \frac{1}{(x-4)^2}$



$y = x(x+3)$

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GRAPHING

II. SYMMETRY

Symmetry helps in graphing. If you have it (you don't always), then you need only graph half the curve - the other half can be mirrored from the first half.

Symmetry about the y-axis occurs when x and $-x$ give the same y-value.

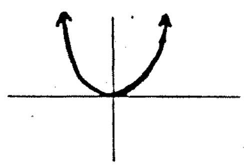
Even Symmetry : $f(-x) = f(x)$

Examples:

$y = x^2$

for $x=2$, $y=4$

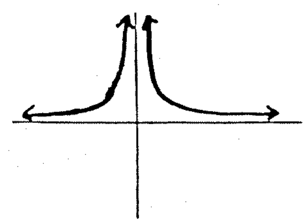
for $x=-2$, $y=4$



$y = \frac{1}{x^2}$

for $x=2$, $y = \frac{1}{4}$

for $x= -2$, $y = \frac{1}{4}$



Symmetry about the origin occurs when x yields y and $-x$ yields $-y$.

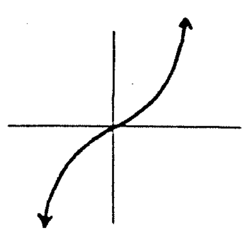
Odd Symmetry : $f(-x) = -f(x)$

Examples:

$y = x^3$

for $x=2$, $y=8$

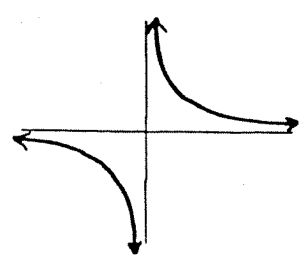
for $x=-2$, $y=-8$



$y = \frac{1}{x}$

for $x=2$, $y = \frac{1}{2}$

for $x=-2$, $y = -\frac{1}{2}$



Some functions have other symmetries, which are not discussed here (see exercise #8); others have no symmetry at all (see exercise #9).

<u>Exercises</u>: By substitution of $-x$ for x (or by a sketch), decide whether the following functions are symmetrical about the y-axis or the origin, or have neither symmetry.

1. $y = 3x^3 + x$

2. $y = x^6 + 2x^2 + 3$

3. $y = 3x^5 + 2x^2 + 1$

4. $y = \sin x$

5. $y = \cos x$

6. $y = \sin x + \cos x$

7. $y = \sqrt{x}$

8. $y = (x+2)^2$

9. $y = 2^x$

<u>Answers</u>: 1. Symm. about origin 2. Symm. about y-axis 3. neither 4. Symm. about origin 5. Symm. about y-axis 6. neither 7. neither (this function is only defined for positive x and positive y). 8. neither (although this function <u>is</u> symmetric about the line $x = -2$). 9. neither.

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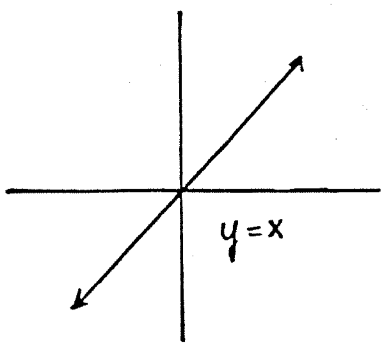
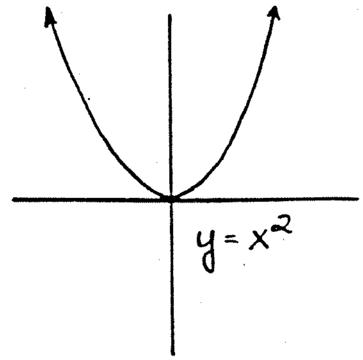
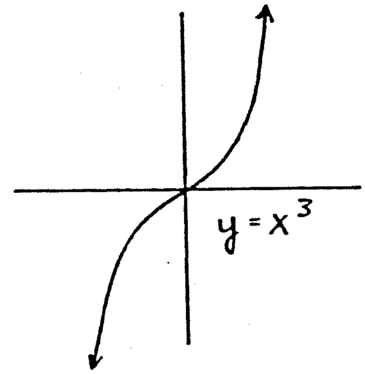
Mathematics Support Capsules

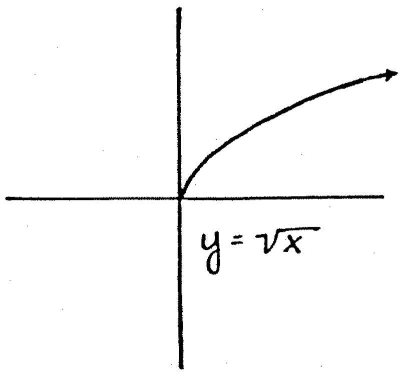
GRAPHING

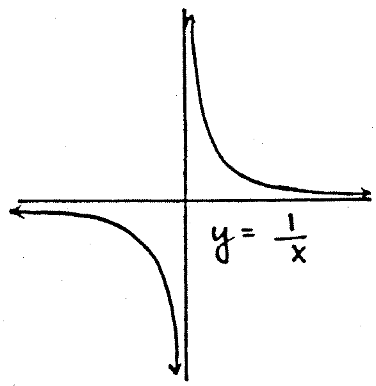
III. GRAPHS YOU SHOULD KNOW

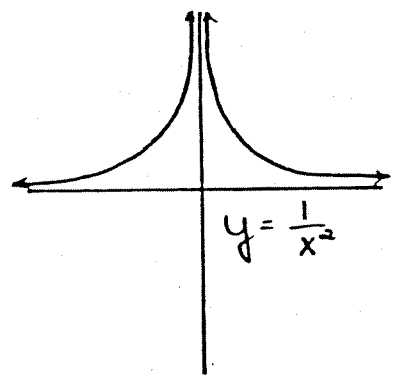


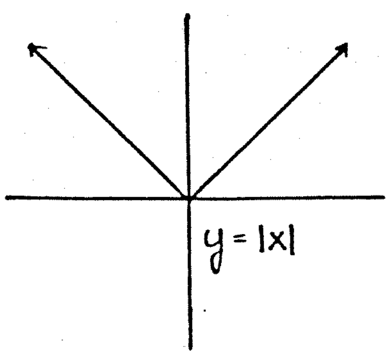


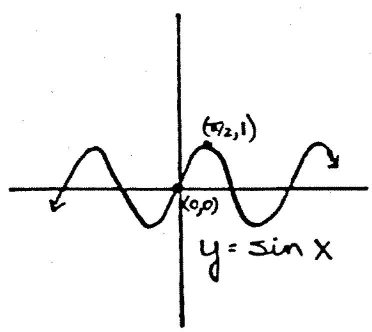


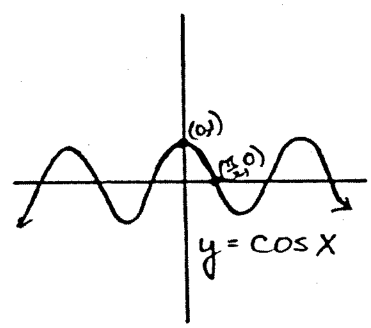


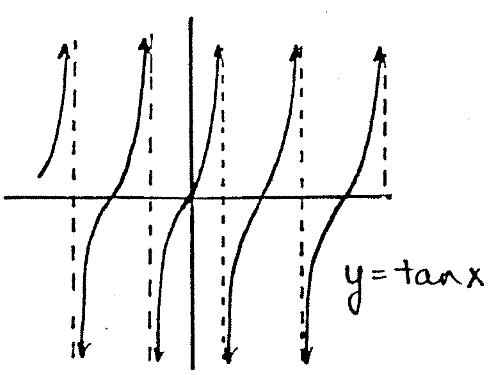


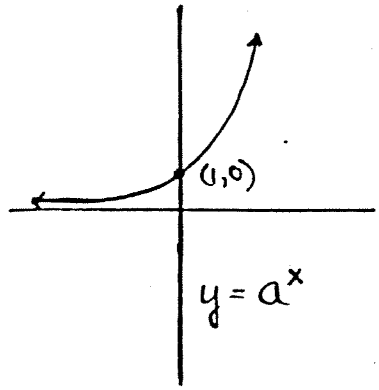


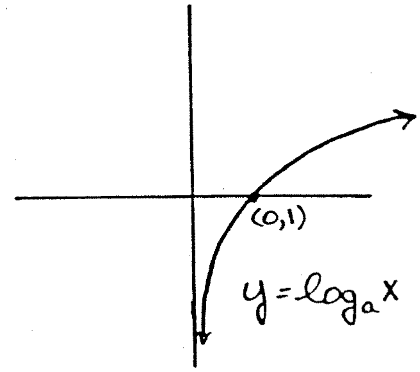








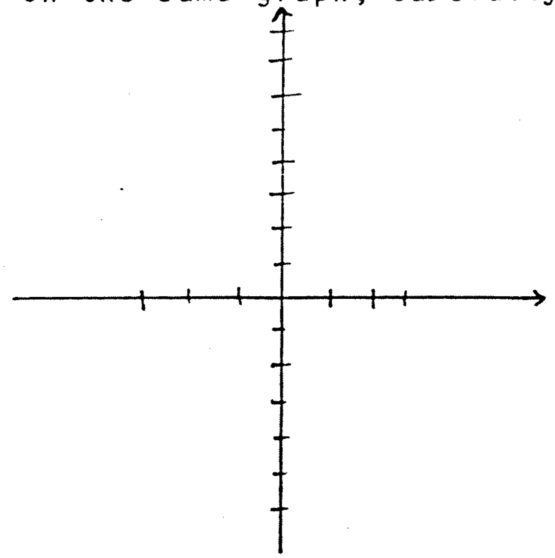




Graph the following 2 functions on the same graph, carefully showing how they intersect.

$y = x^2$

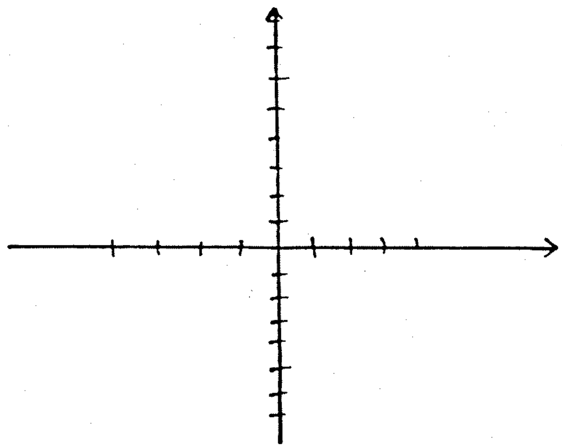
$y = x^4$



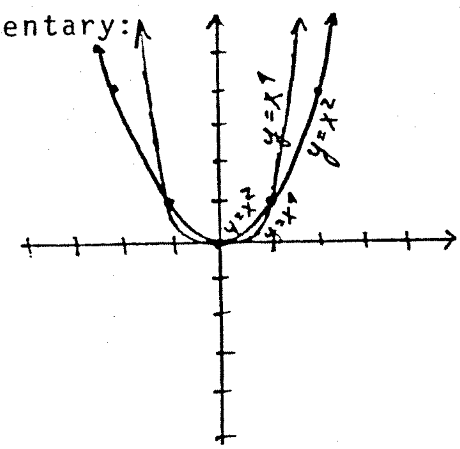
Graph the following 2 functions on the same graph, carefully showing how they intersect.

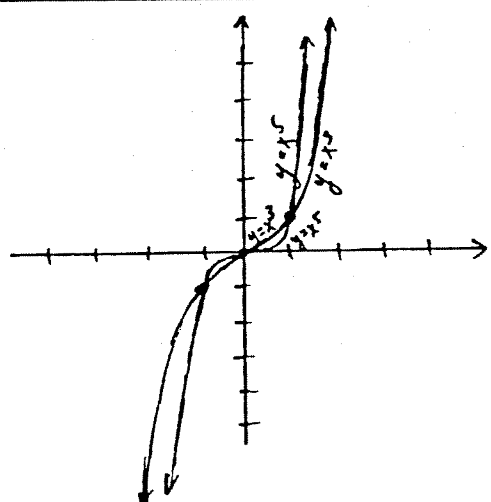
$y = x^3$

$y = x^5$



Commentary:





for <u>even</u> powers of x ($y = x^2$, $y = x^4$, $y = x^6$ ...) the graphs all pass through (0,0), (1,1), and (-1,1). However, the higher the power, the flatter the graph in the middle.

for <u>odd</u> powers of x ($y = x$, $y = x^3$, $y = x^5$, ...) the graphs all pass through (0,0), (1,1), and (-1,-1). However, the higher the power, the flatter the graph in the middle.

Mathematics Support Capsules

GRAPHING

IV. NEW GRAPHS FROM OLD

Suppose we know the graph of $y = f(x)$ then we can very quickly obtain the graphs of a host of related functions.

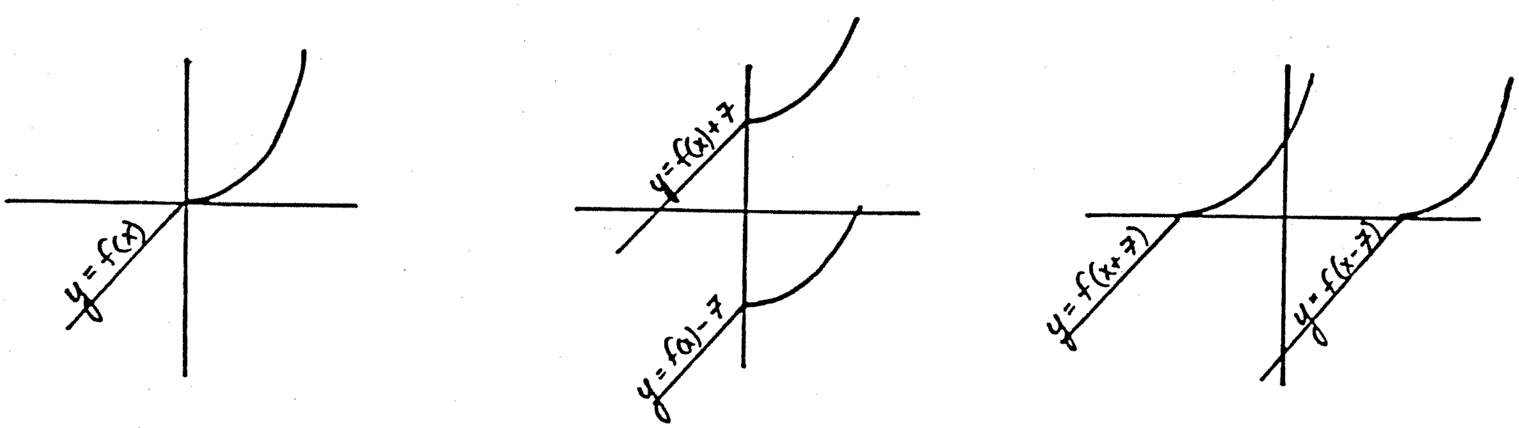
If you <u>add</u> a constant to x or y, the graph <u>slides</u> to a new position.

If a function $g(x)$ is related to $f(x)$ by a relation like the following:

$$g(x) = f(x) + 7 \qquad g(x) = f(x+7)$$
$$g(x) = f(x) - 7 \qquad g(x) = f(x-7)$$

then the graph of $y = g(x)$ can be obtained from the old graph as follows:

If $g(x) =$	DO THIS TO GET THE GRAPH OF $y = g(x)$
$f(x) + 7$	Slide the graph of $y = f(x)$ <u>up</u> 7 units
$f(x) - 7$	Slide the graph of $y = f(x)$ <u>down</u> 7 units
$f(x + 7)$	Slide the graph of $y = f(x)$ to the <u>left</u> 7 units
$f(x - 7)$	Slide the graph of $y = f(x)$ to the <u>right</u> 7 units

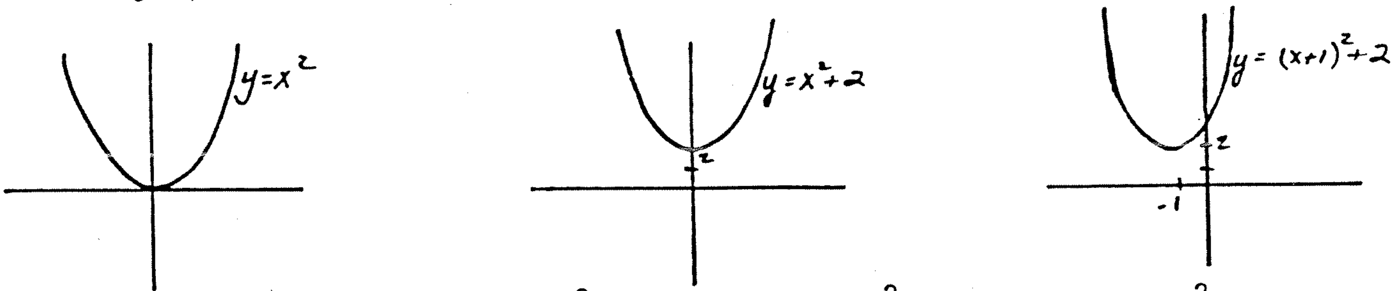


(There is nothing special about 7. It's just used as a "typical number".)

If you forget which way the graph slides, look at $x = 0$.

(Notice that when the constant is <u>inside</u> the parentheses, the graph slides in the <u>opposite</u> direction from what you might expect.)

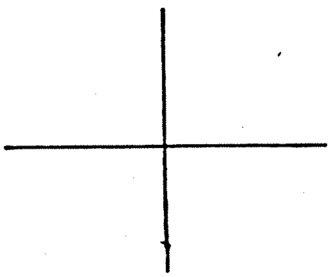
Example: A quick way to graph $y = (x+1)^2 + 2$ is to start with the famous graph of $y = x^2$ and make appropriate slides.



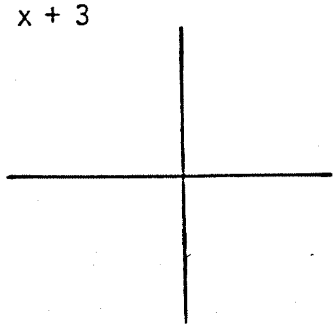
(You could also go from $y = x^2$ to $y = (x+1)^2$ to $y = (x+1)^2 + 2$.)

Try these:

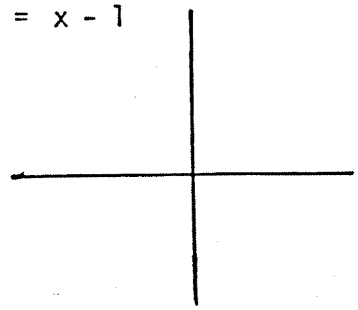
1. $y = x$



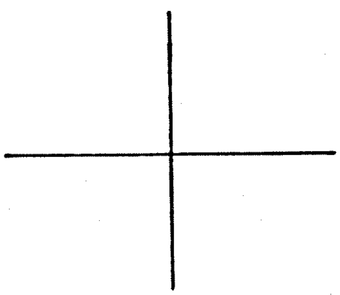
$y = x + 3$



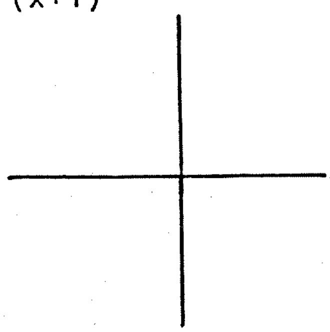
$y = x - 1$



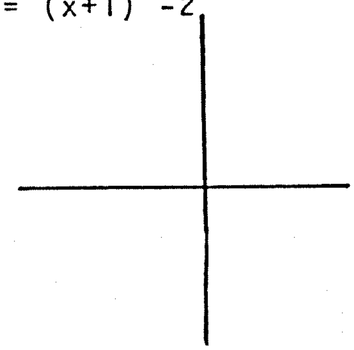
2. $y = x^3 + 1$



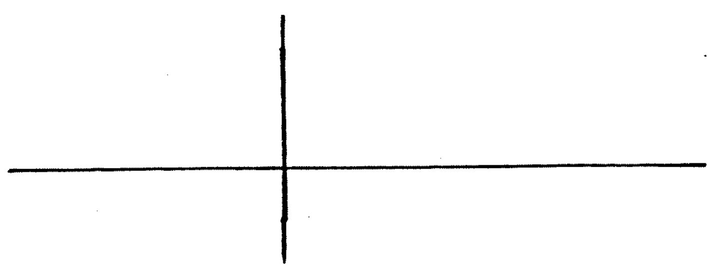
$y = (x+1)^3$



$y = (x+1)^3 - 2$



3. $y = \sin(x + \pi/2)$



What trig identity does this illustrate?

Solutions on p.IV.

If you <u>multiply</u> x or y by a constant, the graph <u>shrinks</u> or <u>stretches</u>. <u>If</u> the constant is <u>negative</u>, the graph also <u>reflects</u>

If a function $g(x)$ is related to $f(x)$ by a relation like the following:

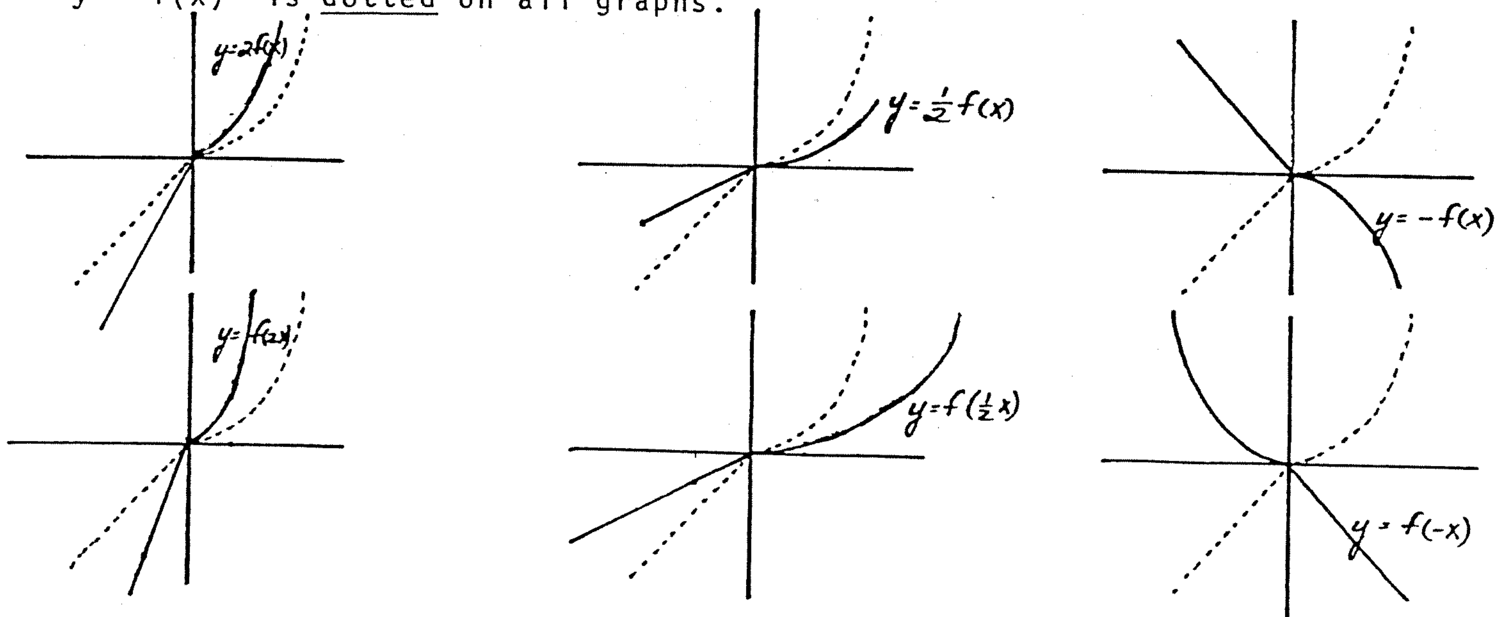
$$g(x) = 2f(x) \qquad g(x) = \frac{1}{2}f(x) \qquad g(x) = -f(x)$$

$$g(x) = f(2x) \qquad g(x) = f(\tfrac{1}{2}x) \qquad g(x) = f(-x)$$

then the graph of $y = g(x)$ can be gotten from the old graph as follows:

If $g(x) =$	DO THIS TO GET THE GRAPH OF $y = g(x)$
$2f(x)$	<u>Stretch</u> the graph of $f(x)$ <u>vertically</u> by a factor of 2
$f(2x)$	<u>Shrink</u> the graph of $f(x)$ <u>horizontally</u> by a factor of 2
$\frac{1}{2}f(x)$	<u>Shrink</u> the graph of $f(x)$ <u>vertically</u> by a factor of 2
$f(\frac{1}{2}x)$	<u>Stretch</u> the graph of $f(x)$ horizontally by a factor of 2
$-f(x)$	<u>Reflect</u> the graph of $y = f(x)$ across the <u>x-axis</u> (i.e., interchange up and down)
$f(-x)$	<u>Reflect</u> the graph of $y = f(x)$ across the <u>y-axis</u> (i.e., interchange left and right)

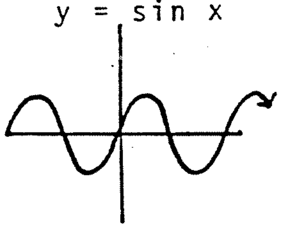
$y = f(x)$ is <u>dotted</u> on all graphs.



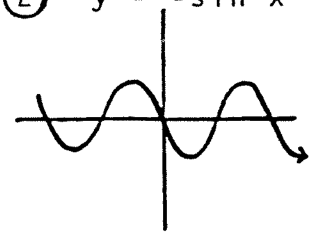
Good test points are $x = 0$ and $x = 1$.

Example: A quick way to graph $y = -2 \sin x$ is to start with the famous graph of $y = \sin x$, reflect it to take care of the $-$ sign, and stretch or shrink it to take care of the factor of 2.

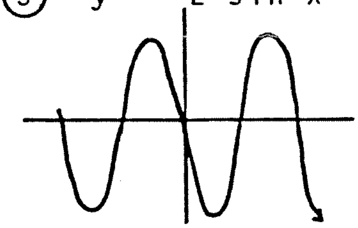
1) $y = \sin x$



2) $y = -\sin x$

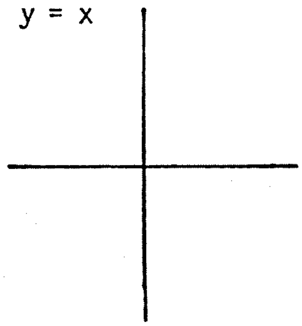


3) $y = -2 \sin x$

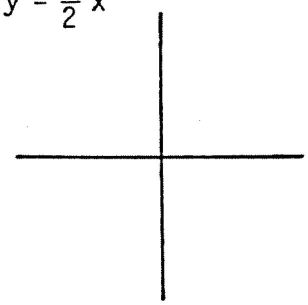


Try these:

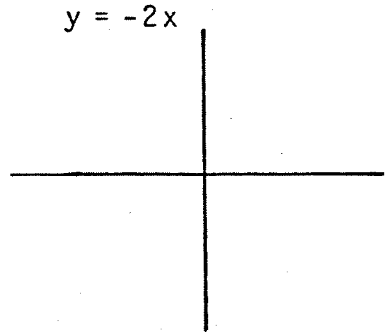
4) $y = x$



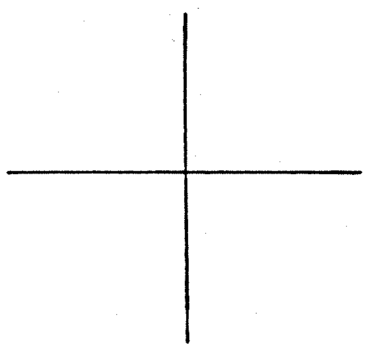
$y = \frac{1}{2} x$



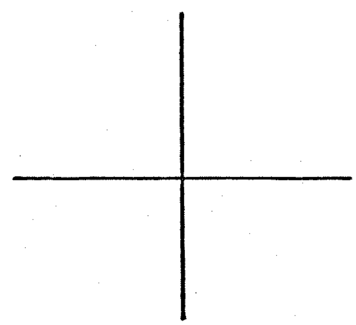
$y = -2x$



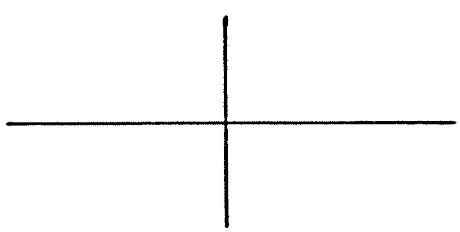
5) $y = -3x^2$



6) $y = 2x^2 + 1$



7) $y = -\sin 2x$

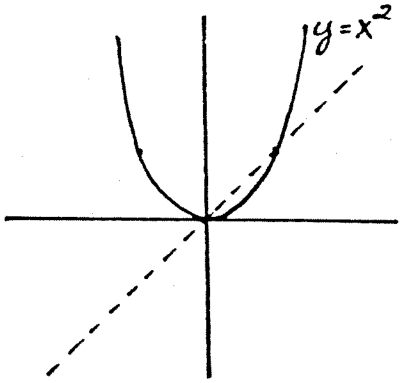


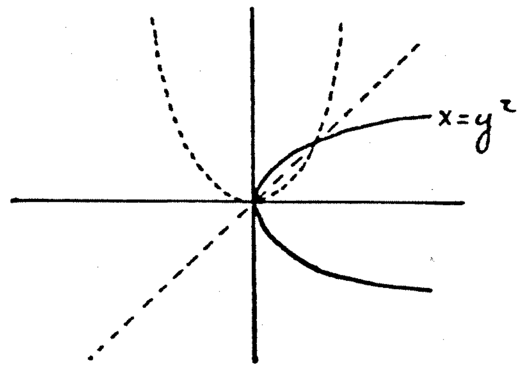
Solutions on p.IV.

If you interchange x and y, the graph reflects about the diagonal line $y = x$.

If $y = f(x)$, then to get the graph of $x = f(y)$, dot in the diagonal $y = x$ and reflect about it.

Example:

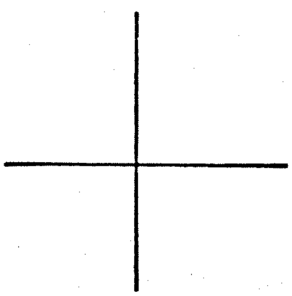




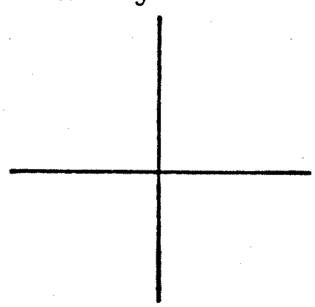
You can fold your paper along $x = y$ and see that one curve is a reflection of the other

Try these:

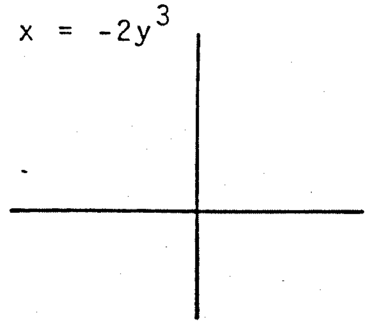
8. $x = y$



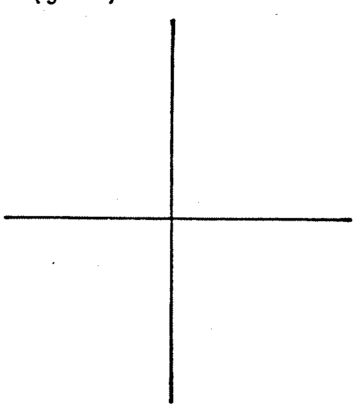
9. $x = y^3$



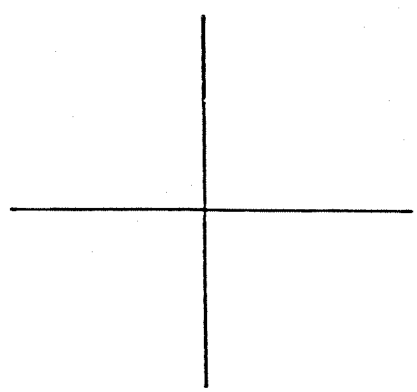
10. $x = -2y^3$



11. $x = (y+2)^2$



12. $x = \sin y$

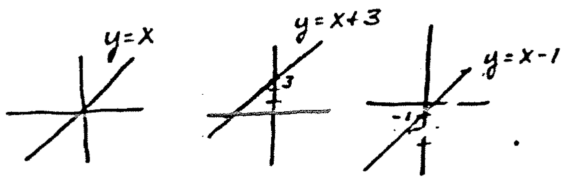


Solutions on p.IV-6

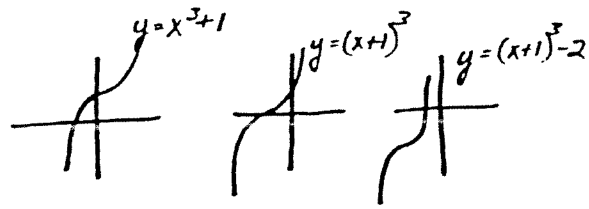
Solutions to Exercises:

p.IV-2:

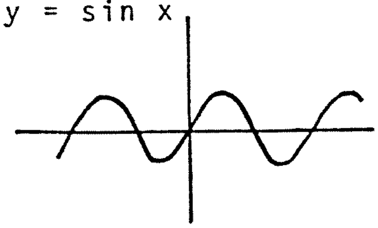
1.



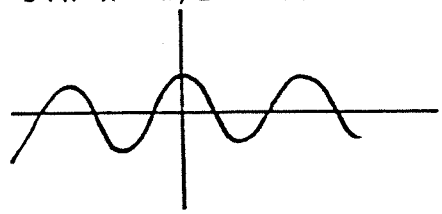
2.



3. y = sin x

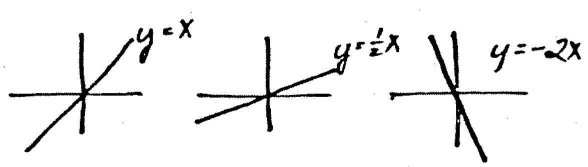


y = sin x + π/2 = cos x

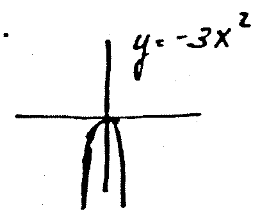


p.IV-4:

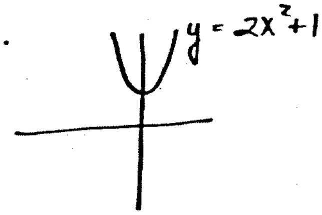
4.



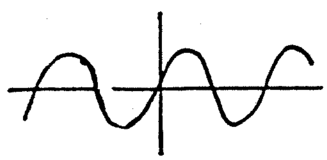
5.



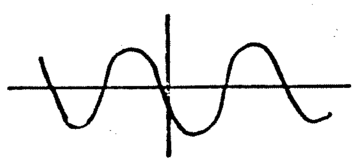
6.



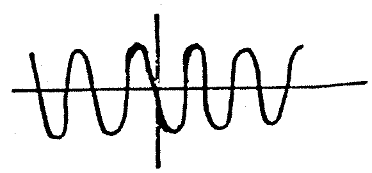
7. y = sin x



y = -sin x

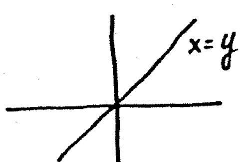


y = -sin 2x



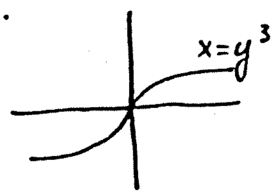
p.IV-5:

8.

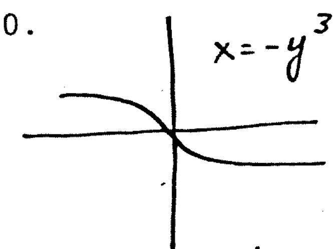


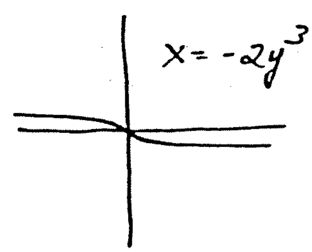
no change - why?

9.

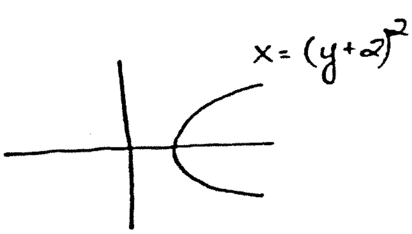


10.

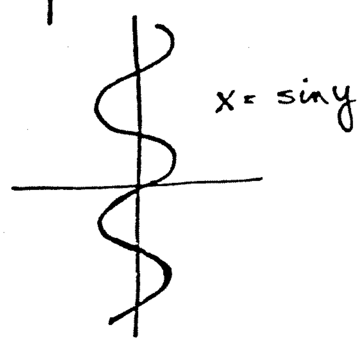




11.



12.



These Mathematics Support Capsules were prepared under the supervision of Beverly West, Lecturer, Department of Mathematics, Cornell University, Ithaca, New York 14853, with assistance from King Chan, Ken Gardner, Kristen Jackson, Ann Michel, Ken Okamoto, and funding from the Exxon Corporation. Reproduction of these items for any commercial purpose is expressly prohibited. The authors solicit your comments, corrections, and suggestions for future revision. August, 1981

Mathematics Support Capsules

GRAPHING

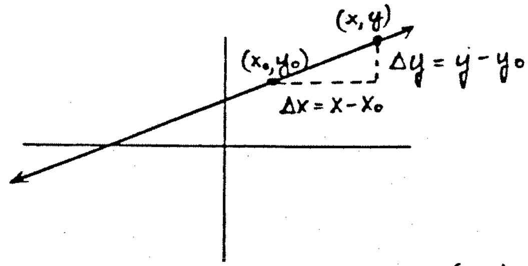
V. STRAIGHT LINES

<u>Slope</u> is the defining characteristic of a line.

$$\boxed{\text{slope} \quad m \equiv \frac{\Delta y}{\Delta x} = \frac{\text{vertical change}}{\text{corresponding horizontal change}}}$$

A particular line is tied down completely if you know just one more bit of information, such as

a point on the line

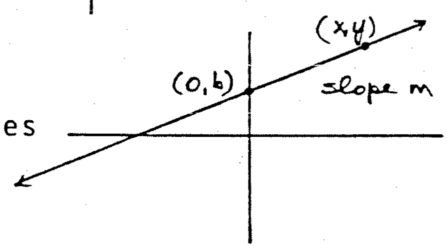


$$\boxed{m = \frac{y - y_0}{x - x_0}}$$

or,

the y-intercept b

(the point where the line crosses the y-axis, where x=0)



$$\boxed{y = mx + b}$$

The two boxed equations are equivalent. Why? (Hint: multiply out the first and get it in the form of the second.)

<u>Any</u> equation of the form ax + by + c = 0 is a <u>linear</u> equation that has a straight line graph.

(Show that you can arrange it in the form y = mx + b).

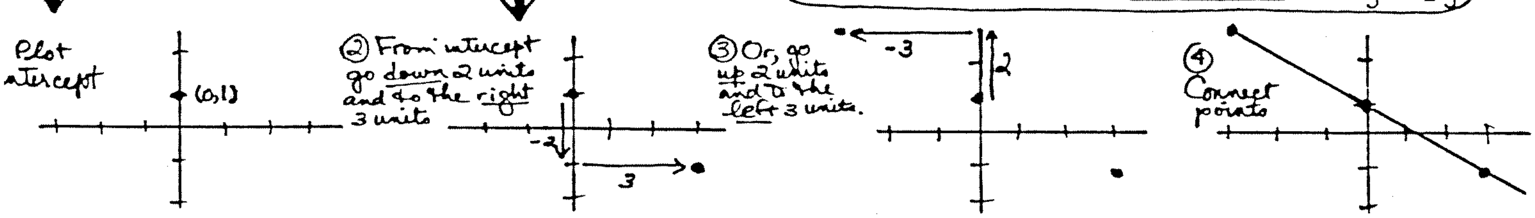
Example: Graph 3y = -2x + 3 .

You can rewrite this as $y = \overbrace{-\frac{2}{3}}^{m}x \overbrace{+ 1}^{b}$

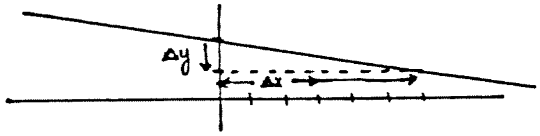
b=1, so the point (0,1) is on the graph

$m = -\frac{2}{3} = \frac{\text{for every 2 units y moves,}}{\text{x moves 3 units in the } \underline{\text{opposite}} \text{ direction}}$

because slope is <u>negative</u> $= \frac{-2}{3} = \frac{2}{-3}$



Example: From the following graph, determine the equation of the line:



① Draw a right triangle from the line to determine its slope

② Count units: $\frac{\Delta y}{\Delta x} = \frac{1 \text{ unit DOWN}}{5 \text{ units RIGHT}} = \frac{\text{negative}}{\text{positive}}$ slope $= -\frac{1}{5}$

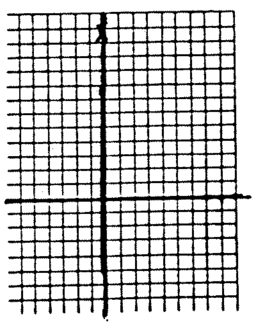
③ read y - intercept from graph: here = (0,2)

equation = $y = -\frac{1}{5}x + 2$

Exercises:

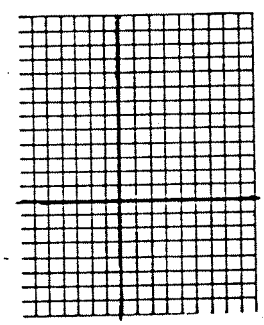
1. Plot on the same axes the following lines:

$y=x$ $y=-x$
$y=2x$ $y=0$
$y=3x$ $x=0$

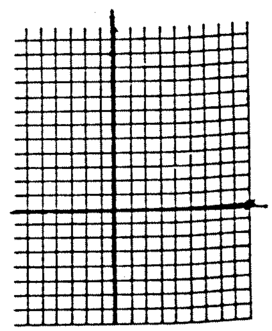


2. Plot on the same axes the following lines:

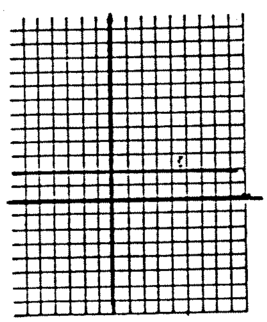
$y=2x+5$ $y=-\frac{1}{2}x + \frac{9}{2}$
$y=2x-3$



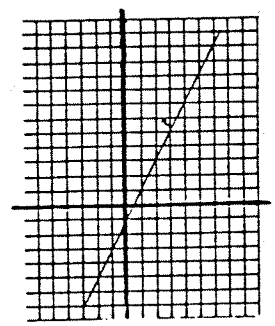
3. Find the equation of the line through (2.3) and (-1,-2) [Hint: find slope between 2 points] and graph the line.



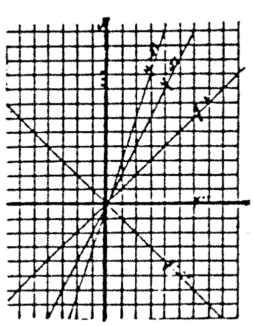
4. Find the equation of this line:

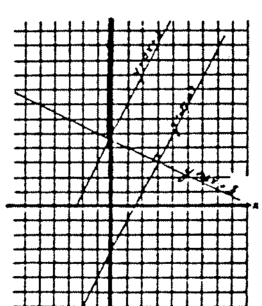


5. Find the equation of this line:

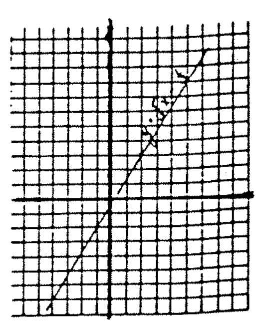


Solutions:





note: lines with the same slope are parallel. Perpendicular lines have slopes that are negative reciprocals.



$y = \frac{5}{3}x - \frac{1}{3}$

4. $y = 2$

5. $y = -2x-1$

Mathematics Support Capsules

GRAPHING

VI. POLYNOMIALS

We have on reserve in the Mathematics Library and in the Mathematics Support Center a paper by V. Frederick Rickey of Bowling Green State University on Qualitative Graphing Techniques. It presents some nice ideas that are not usually taught.

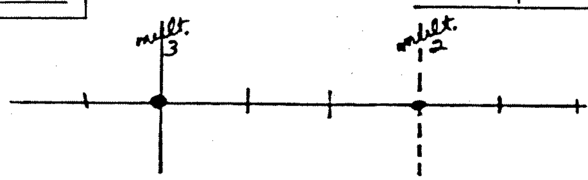
Here we give only a brief introduction; we recommend that you look at Rickey's work for further study.

We shall examine $y = f(x)$ where $f(x)$ is a polynomial that can be factored into linear factors. Not all polynomials can be factored into linear factors. Rickey discusses only those that can, and we are further restricting the present discussion to those linear factors of the form $x+a$ or $x-a$, so $f(x) = (x+a)(x+b)\ldots$.

The following four steps lead to a qualitative picture that gives much of the essential information about the graph without painful point-plotting; this is especially useful for a polynomial of high degree.

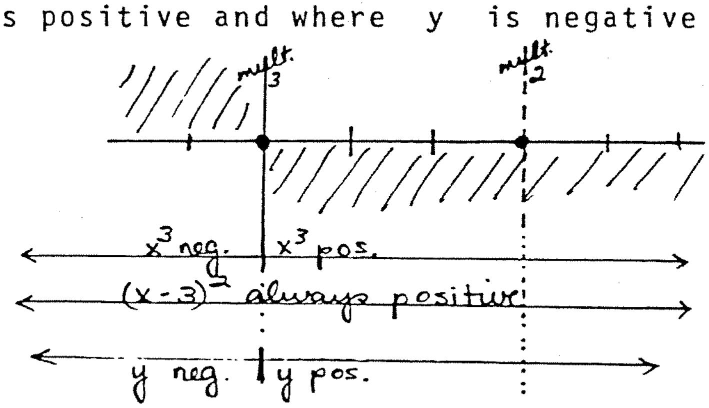
Example: $y = x^5 - 6x^4 + 9x^3 = x^3(x^2 - 6x + 9) = x^3(x-3)^2$

$\boxed{1 \quad \text{Locate the zeros}}$ and note their multiplicity (power of the factor)



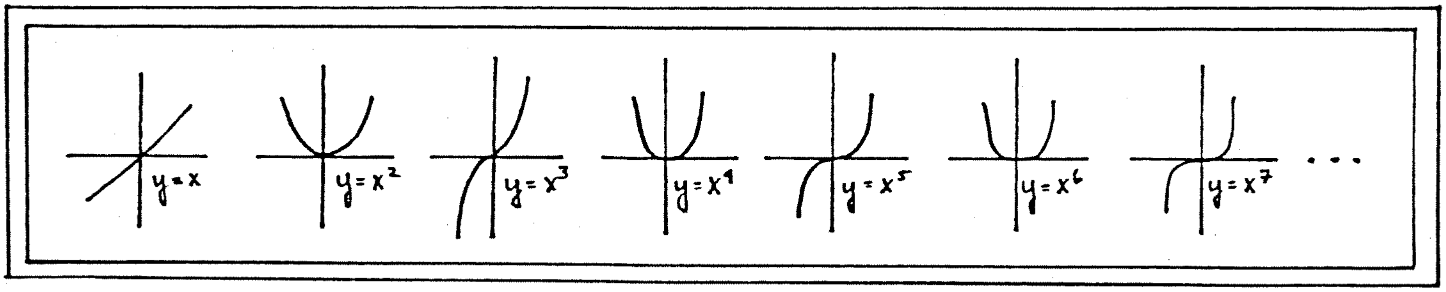
$\boxed{2 \quad \text{Determine the excluded regions}}$ by using the factors to determine

where y is positive and where y is negative (See Capsule I.)



$\boxed{\text{3 Sketch the } \underline{\text{shape}} \text{ of the graph } \underline{\text{at the zeros}}\text{.}}$ The shape is determined by the multiplicity of the zeros.

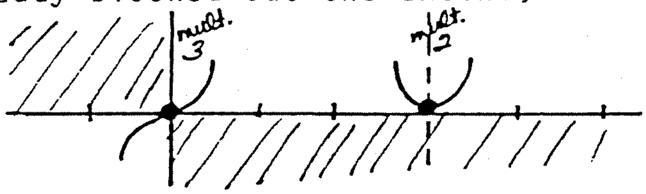
Recall (See Capsule III) that



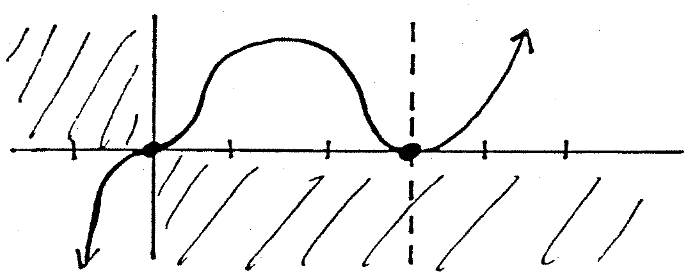
At $x=0$, our graph has the shape of x^3, because $y=x^{3}(x-2)^2$

At $x=2$, our graph has the shape of x^2, because $y=x^3(x-2)^{2}$

Since we've already blocked out the extent, we can easily fit in these shapes.



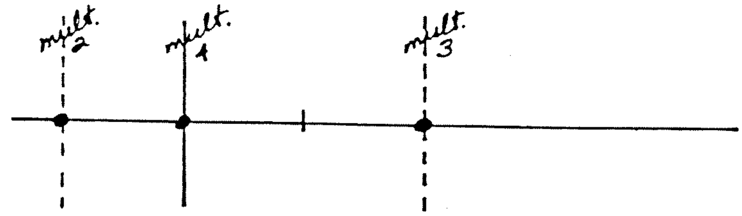
$\boxed{\text{4 Connect these pieces smoothly.}}$ Where forced, you must put a hump (as between the zeroes in this example), but there will be <u>no</u> extra humps or bends in the graph of the function.



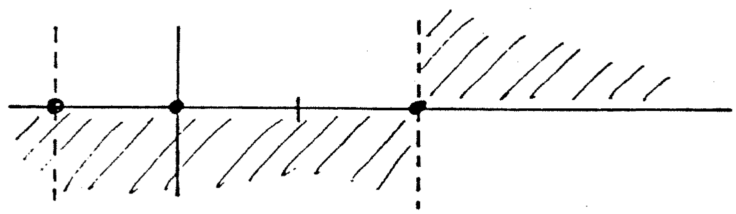
You now have a fine qualitative picture of $y = x^3(x-2)^2$. With calculus you could find out exactly where the humps and bends occur - that quantitative study is beyond the scope of this discussion.

Another example: $f(x) = -(x+1)^2 (x-2)^3 \; x^4$

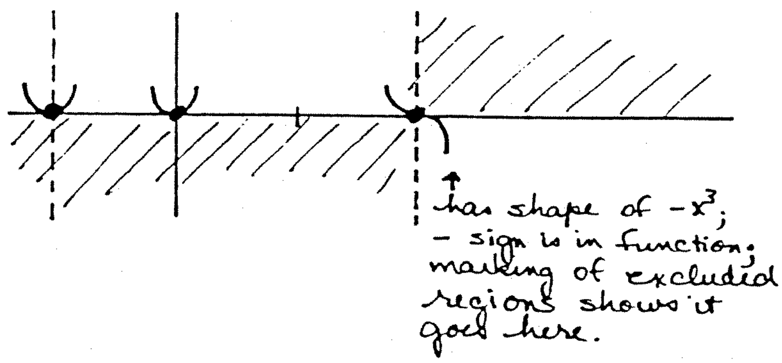
1 zeroes, with multiplicity



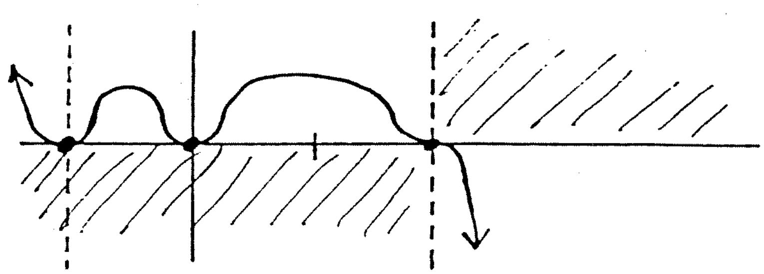
2 excluded regions



3 shape at zeroes



4 connect



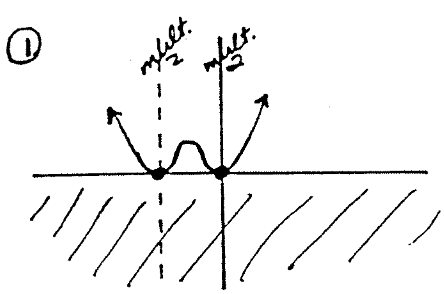
Exercises:

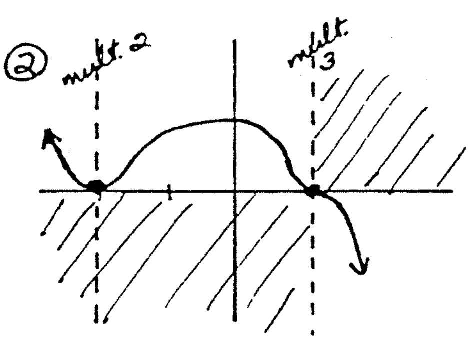
1. $y = x^2(x^2+2x+1) = x^2(x+1)^2$

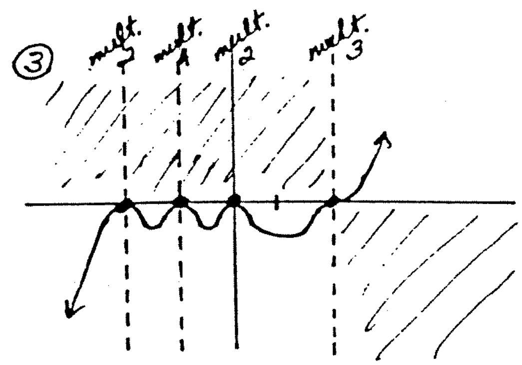
2. $y = -(x-1)^3(x+2)^2$

3. $y = (x+1)^4(x-2)^3(x+2)^2\ x^2$

Solutions:







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Mathematics Support Capsules

GRAPHING

VII. FINDING ASYMPTOTES

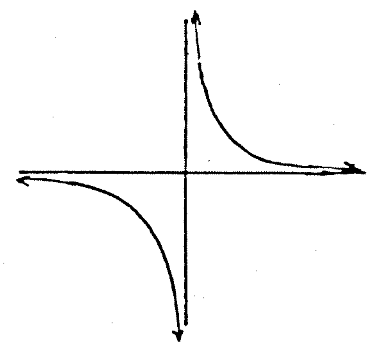
<u>Asymptotes</u> are lines that a graph approaches but never touches.

<u>Horizontal asymptotes</u> may occur when $x \to \infty$ or $x \to -\infty$, if $y \to$ constant.

Example: $y = \frac{1}{x}$

As $x \to \infty$, $y \to 0$ (and is positive)
(as $x = 1, 2, 3, 4, 5 \ldots \to \infty$,
$\frac{1}{x} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \to 0$) and $y = 0$
is a horizontal asymptote.

As $x \to -\infty$, $y \to 0$ (and is negative)
(say = 0 is a horizontal asymptote)

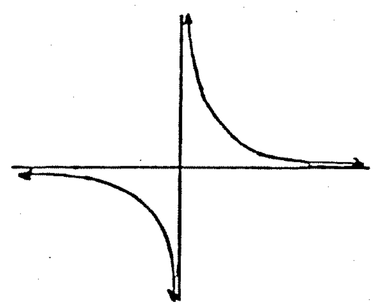


<u>Vertical asymptotes</u> occur whenever a denominator $\to 0$.

In a rational function (that is, a quotient of polynomials) there is a vertical asymptote for every value of x that makes the denominator zero. The graph cannot cross these vertical asymptote; these are places where a sudden jump is often made in the graph.

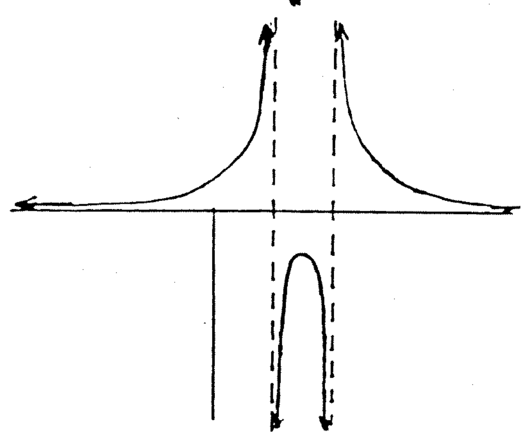
Example: $y = \frac{1}{x}$

As $x \to 0$, denominator $\to 0$
(so there is a vertical asymptote at $x = 0$)



Example: $y = \frac{1}{(x-1)(x-2)}$

As $x \to 1$, denominator $\to 0$
As $x \to 2$, denominator $\to 0$
(so there are vertical asymptotes at $x = 1, x = 2$)
As $x \to \infty$, $y \to 0$ (and is positive)
As $x \to -\infty$, $y \to 0$ (and is positive)
(so $y = 0$ is a horizontal asymptote).



Find the asymptotes for these functions:

1) $y = \dfrac{1}{x^2-5x+6}$ Vertical:
Horizontal:

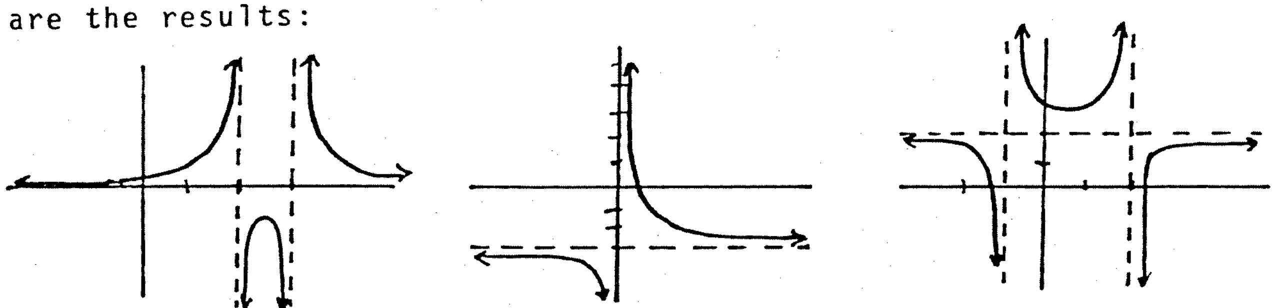
2) $y + 3 = 1/x$ Vertical:
Horizontal:

3) $y - 2 = \dfrac{-1}{(x+1)(x-2)}$ Vertical:
Horizontal:

Answers: (1) Vert: $x = 3, x = 2$, Horiz: $y = 0$; (2) Vert: $x = 0$, Horiz: $y = -3$; (3) Vert: $x = -1, x = 2$, Horiz: $y = 2$

You can sketch the graphs now, by determining excluded regions and a little judicious point plotting.

Here are the results:



For additional techniques of graphing rational functions, see V.Frederick Rickey, Qualitative Graphing Techniques, pp.26-62. On reserve in Mathematics Library and Mathematics Support Center.

There also exist oblique asymptotes (between horizontal and vertical) which haven't been discussed here. See Deborah Hughes-Hallett, The Math Workshop: Elementary Functions, pp.79-90.

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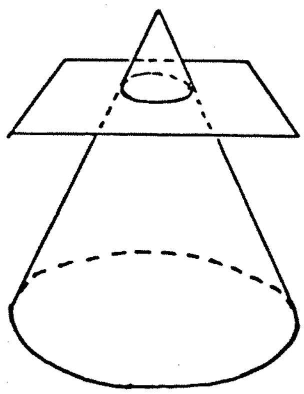
Mathematics Support Capsules

GRAPHING

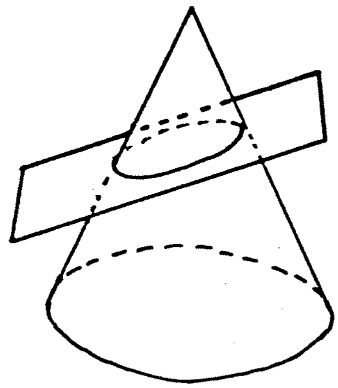
VIII. CONIC SECTIONS

The conic sections are the figures made by a plane intersecting a cone.

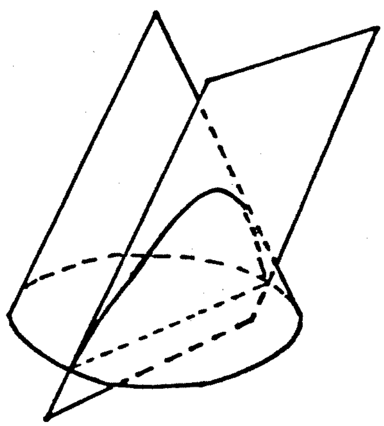
circle



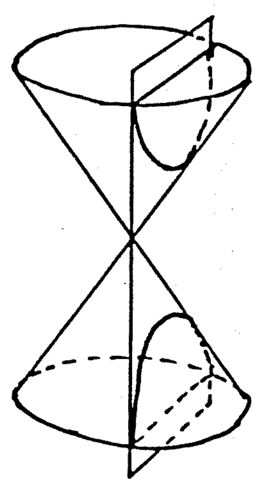
ellipse



parabola



hyperbola



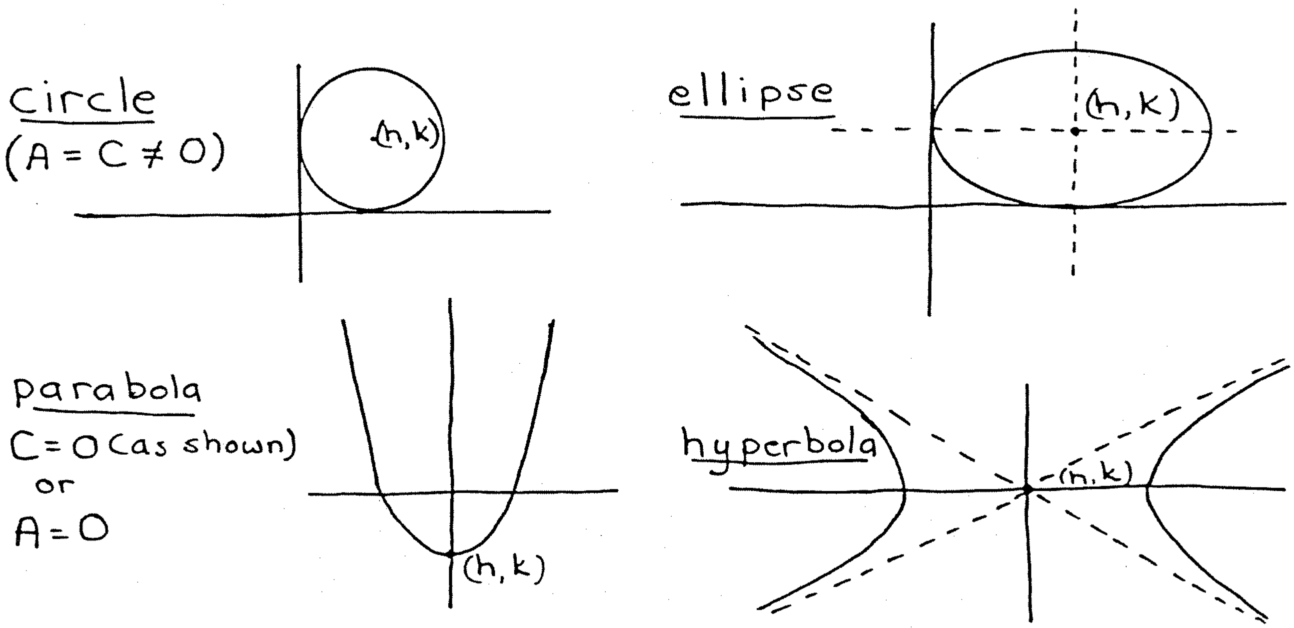
We have a wooden model in the Mathematics Support Center that can be taken apart to show these sections. Ask to see it.

Every conic section has an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ and every equation of this form has a graph which is a conic section.

So, whenever you have a relation with only quadratic terms, you can be sure that its graph is a conic section.

$$\boxed{Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0}$$

If $B = 0$, life is simplest. Then each conic section has a standard form with "center" at (h,k) and axes parallel to the x- and y-axes.



Details of the standard form for each conic section are on the following pages.

Getting into standard form requires from algebra the technique called "completing the square", which is demonstrated in the examples. If you need review of this technique, see

Capsule "Completing the Square"

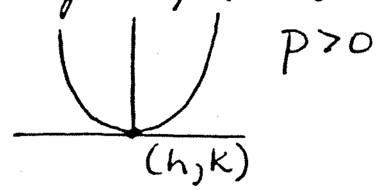
or

D.Hughes-Hallett, The Math Workshop: Elementary Functions, Ch.5

(For $B \neq 0$, see p.VIII-7)

PARABOLA:

The standard form of the parabola: $y-k=p(x-h)^2$
vertex at (h,k)
If $p>0$: opens upwards
$p<0$: opens downwards



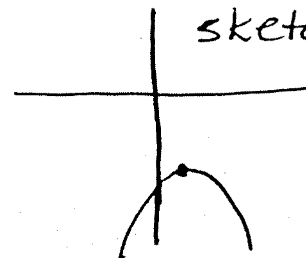
Other forms of the parabola are:

$y=ax^2+bx+c$ This can be transformed into standard form by using the method of completing the square.

$x=ey^2+fy+h$ This is a parabola opening sideways. Again, completing the square would put it in standard form.

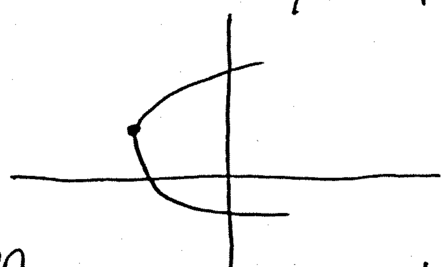
sketch

Examples: (1) $y=-3(x-2)^2-6$, $y+6=-3(x-2)^2$ } vertex: $(2,-6)$, opens down



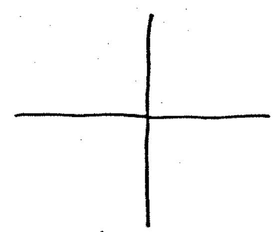
(2) $x=y^2-2y-1$
$x+1=y^2-2y$
$x+1+1=y^2-2y+1$
$x+2=(y-1)^2$

vertex: $(-2,1)$
opens: to right

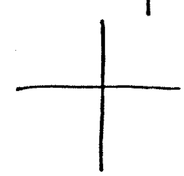


Find the vertex and opening direction.
Also sketch

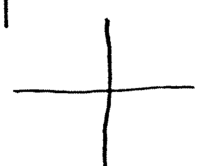
1) $y=\frac{1}{2}(x+1)^2+2$
vertex:
opens:



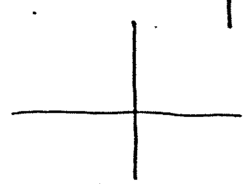
2). $y=x^2-4x+2$
vertex:
opens:



3) $x=2(y-3)^2+3$
vertex:
opens:



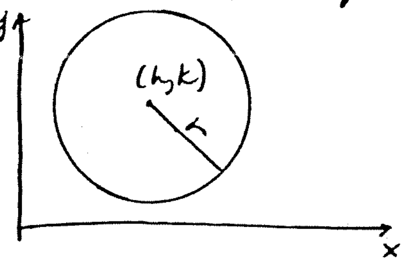
4) $x=-y^2+4y+2$
vertex:
opens:



nswers: (1) $(-1,2)$, up (2) $(2,-2)$, up (3) $(3,3)$ opens to right (4) $(6,2)$ opens to left

CIRCLE:

The standard form of a circle: $(x-h)^2+(y-k)^2=r^2$
Center at (h,k)
radius: r units



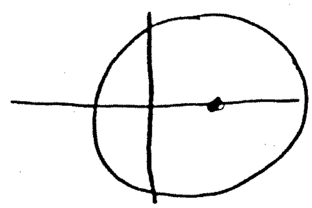
Another form of the circle:
$y^2+by+x^2+cx+d=0$

The x^2 and y^2 must have the same coefficient. Put into standard form by completing square.

Examples:

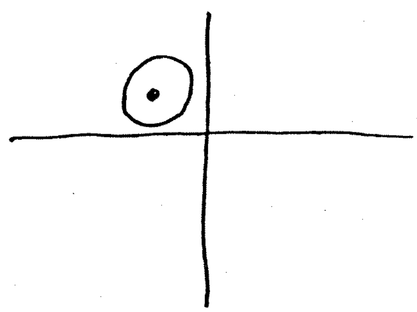
$(x-2)^2+y^2=25$
center at $(2,0)$
radius $=5$

sketch:



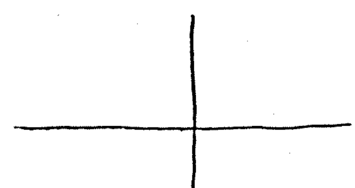
$$9x^2+18x+9y^2-18y+14=0$$
$$9x^2+18x \quad +9y^2-18y \quad =-14$$
$$x^2+2x \quad +y^2-2y \quad =-14/9$$
$$(x^2+2x+1)+(y^2-2y+1) = -14/9+1+1 = 4/9$$
$$(x+1)^2+(y-1)^2 = 4/9 = (2/3)^2$$

center: $(-1,1)$
radius $=2/3$

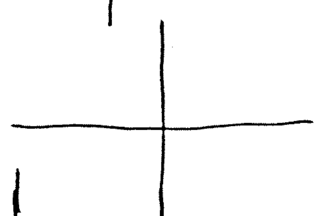


Find the center, the radius, and sketch:

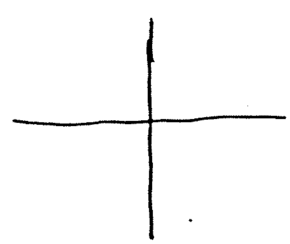
(1) $x^2+(y+4)^2=4$
center:
radius:



(2) $-\frac{1}{2}x^2+2x-\frac{1}{2}y+\frac{9}{2}y=-\frac{3}{2}$
center:
radius:



(3) $3y^2+12y+3x^2-6x=-12$
center:
radius:



Answers: (1) $(0,-4)$, 2 (2) $(2,3)$, 4 (3) $(1,-2)$, 1

ELLIPSE:

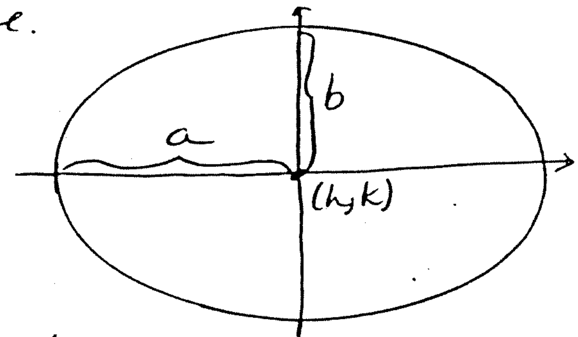
The standard form of an ellipse: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

Center at (h,k)

If $a>b$, the length of the major axis $=2a$

the length of the minor axis $=2b$

otherwise if $a<b$, it is the reverse.



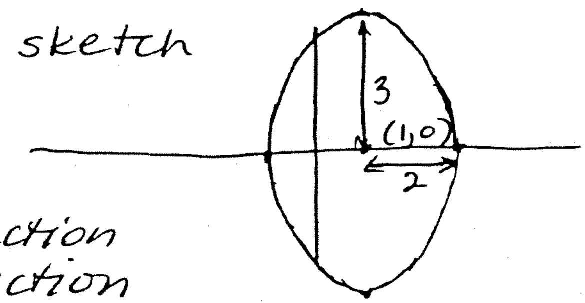
Another form of the ellipse:

$ax^2+bx+cy^2+dy+e=0$

The x^2 and y^2 terms have the same sign. After completing the squares, a positive number must appear on the right hand side. If there isn't a positive number, it is not an ellipse.

Examples:

sketch



(1) $\frac{(x-1)^2}{4}+\frac{(y^2)}{9}=1$

center: $(1,0)$

major axis: length 6 in y direction

minor axis: length 4 in x direction

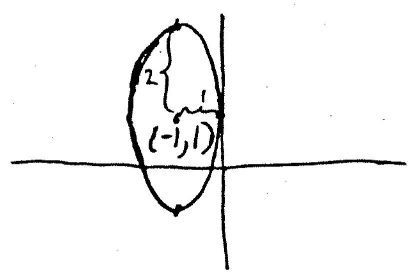
(2) $4x^2+8x+y^2-2y+1=0$

$4(x^2+2x+\quad)+(y^2-2y+\quad)=-1$

$4(x^2+2x+1)+(y^2-2y+1)=-1+4+1$

$4(x+1)^2+(y-1)^2=4$

$(x+1)^2+\frac{(y-1)^2}{4}=1$



center $(-1,1)$ major axis length: 4, minor axis length: 2

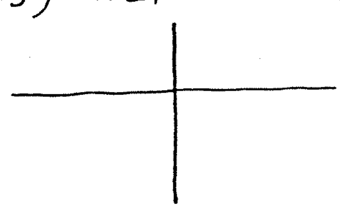
Find the center, lengths of major and minor axes, and sketch:

$4(x+3)^2+(y-2)^2=36$

center:

length major axis:

length minor axis:

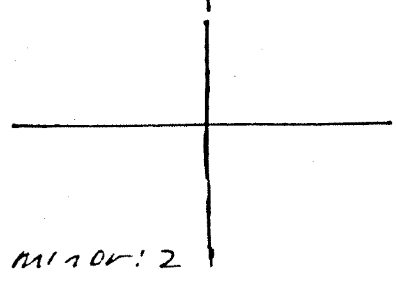


$x^2+2y^2+8y+6=0$

center:

length major axis:

length minor axis:



Answers: (1) (-3,2) major: 12 minor: 6 (2) (0,-2), major: $2\sqrt{2}$ minor: 2

HYPERBOLA:

The standard form of an hyperbola: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

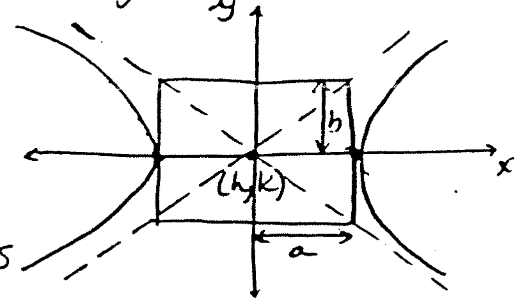
The Center is at (h,k)

Asymptotes are lines with slope $\pm(b/a)$ through the center (h,k)

ther forms of the hyperbola:

$ax^2+bx \quad cy^2+dy+e=0$

by completing squares this can be put into standard form. (Note that the x^2 and y^2 coefficients have different signs)



$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

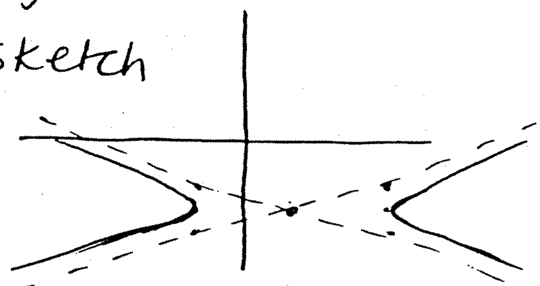
This hyperbola opens vertically.

There are many other forms of the hyperbola which you'll find when you graph them. (They don't reduce to the standard form)

sketch

xamples: (1) $\frac{(x-2)^2}{4^2} - (y+3)^2 = 1$

center: $(2,-3)$, asymptotes' slope $\pm\frac{1}{4}$



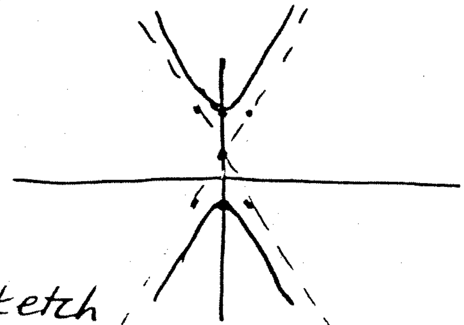
(2) $y^2-2y-2x^2=1$

$(y^2-2y+1) - 2(x^2) = 1+1=2$

$(y-1)^2 - 2(x^2) = 2$

$\frac{(y-1)^2}{2} - x^2 = 1$

center: $(0,1)$; asymptotes' slopes $\pm 2/1$

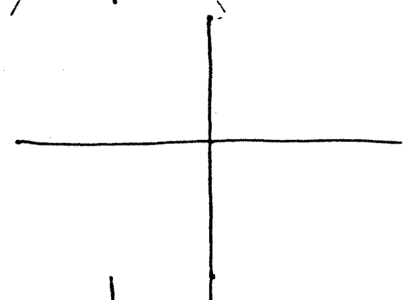


t the following into standard form and sketch,

$9x^2-18x-4y^2-16y=43$

center:

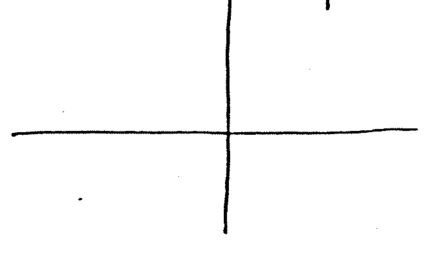
asymptotes' slopes:



) $\frac{(y-1)^2}{36} - \frac{x^2}{9} = 1$

center:

asymptotes' slopes:



NSWERS: (1) $(1,-2)$, $\pm 3/2$; (2) $(0,1)$, ± 2

Now, back to the general form for any conic:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

If $B \neq 0$, then the graph is <u>rotated</u> from the standard position. The exact angle of rotation can be calculated (see Thomas & Finney, <u>Calculus and Analytic Geometry</u>, pp.426-430, or Swokowski, <u>Calculus with Analytic Geometry</u>, 2nd edition, pp.352-356).

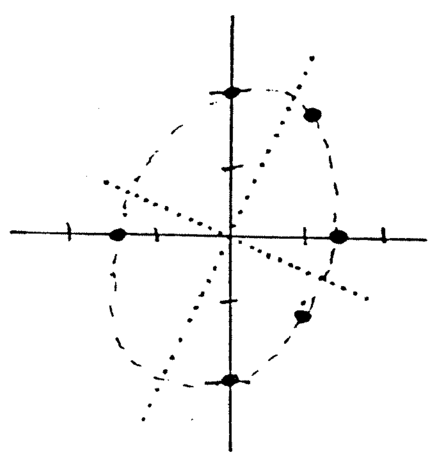
However, in many graphing problems you may have enough information and saavy to figure out the picture without that precise information.

Try $y = 4x^2 - xy + 2y^2 - 8 = 0$.

Solution:

First, plot intercepts $(x = 0, y = 0)$. This gives 4 points, and you will see they cannot lie on a parabola, a hyperbola, or a circle. Since the graph must be a conic section (why?) it must be an ellipse. But the xy term tells us it's a <u>tilted</u> ellipse. If you struggle through <u>one</u> more x-value and plot the y-values, you'll have a pretty nice idea of the ellipse.

x	y
0	± 2
$\pm\sqrt{2} \approx 1.4$	0
1	$\frac{1 \pm \sqrt{33}}{4} \approx \frac{1 \pm 5.7}{4}$ $= 1.7$ or -1.2



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Mathematics Support Capsules

GRAPHING
IX. OVERALL STRATEGY: AN EXAMPLE

SOME BASIC STEPS FOR GRAPHING PROBLEMS:

1) Is it a familiar function? See if it is a conic that can be transformed into standard form. (Capsules III,IV,V,VI,VIII)
2) Determine the intercepts. (Capsule I)
3) Is it symmetrical? (Capsule II)
4) Are there any asymptotes? (Capsule VII)
5) Where is the curve? (What regions are excluded?) (Capsule I)
6) Now can you sketch the curve?

To show how to use these steps, here's an example completely worked out.

Example: Graph $y-1 = \dfrac{1}{x^2-2}$

1) Is it familiar?

$$(y-1)(x^2-2) = 1$$

$yx^2-2y-x^2+2 = 1 \Longrightarrow$ <u>No</u> it isn/t a conic. There is no yx^2 term in the form $ax^2+bx+cxy+dy^2+ey+g = 0$, which covers all conics.

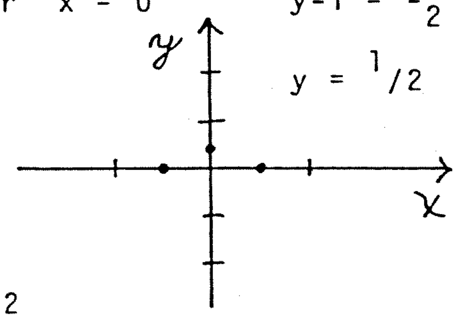
2) Determine the intercepts

For $y = 0$

$$-1 = \frac{1}{(x^2-2)}$$
$$-x^2+2 = 1$$
$$-x^2 = -1$$
$$x^2 = 1$$
$$x = \pm 1$$

For $x = 0$

$$y-1 = -\frac{1}{2}$$
$$y = {}^{1}/2$$



3) Is it symmetrical?

Test by plugging in easy numbers, say ±2

$x=2$

$$y-1 = \frac{1}{(2)^2-2} = \frac{1}{4-2}$$
$$y-1 = {}^{1}/2$$
$$y = {}^{3}/2$$

$x=-2$

$$y-1 = \frac{1}{(-2)^2-2} = \frac{1}{4-2}$$
$$y-1 = {}^{1}/2$$
$$y = {}^{3}/2$$

In fact, $-x$ and x give the same y value for any x, so there is symmetry about the y-axis.

) Are there any asymptotes?

There is a fraction: $\frac{1}{x^2-2}$. So, first check to find where the denominator = 0.

$$\text{Set} \quad x^2-2 = 0$$
$$x^2 = 2$$
$$x^2 = \pm\sqrt{2}$$

There are vertical asymptotes at $x = \pm\sqrt{2} \approx \pm 1.4$

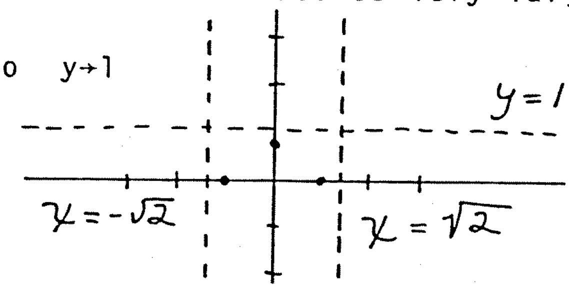
ext: are there any horizontal asymptotes? See what happens as $x \to \infty$ and as $x \to -\infty$

For $1/x^2-2$, as $x \to \infty$ the denominator becomes very large, so $1/x^2-2 \to 0$.

As $x \to -\infty$ the denominator also becomes very large, so $1/x^2-2 \to 0$.

As $1/x^2-2 \to 0$, $y-1 \to 0$, so $y \to 1$

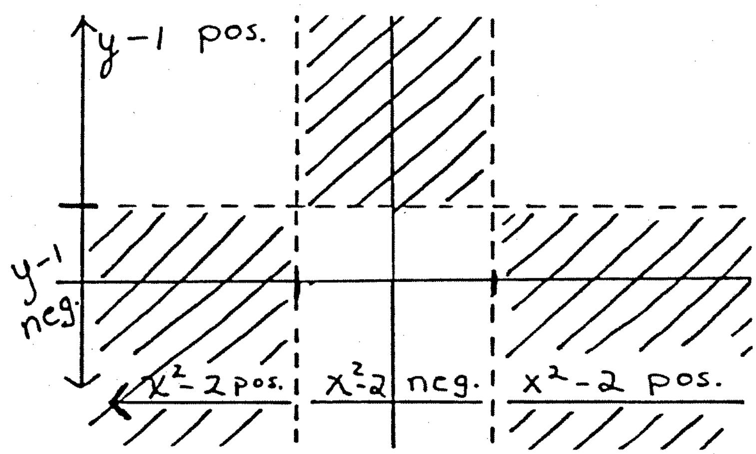
Sketch in the asymptotes:



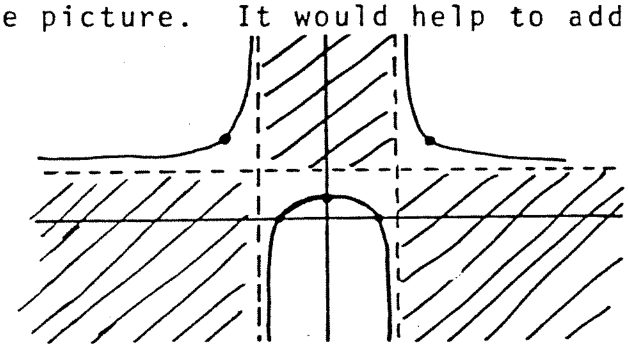
5) Where is the curve?

$$(y-1) = \frac{1}{x^2-2}$$

so $(y-1)$ and (x^2-2) must have the same sign.



6) Now can you sketch the curve? As $x \to \infty$, $y \to 1$. Is it above or below the asymptote $y = 1$? x^2 is always positive; and for $x > \sqrt{2}$, $\frac{1}{x^2-2}$ is positive. So the curve would be above the line $y=1$. By symmetry, it would also be on the left side of the picture. It would help to add the points found in step 3. Then the curve is smooth and can't cross the veritcal asymptotes, so let it approach the vertical asymptotes.



Last, what is happening in the section where the 3 intercepts are? As x gets closer to $\sqrt{2}$, y gets more negative and tends toward $-\infty$.

Further Reference: See D. Hughes-Hallett, Elementary Functions, Chap, 4; V. Frederick Rickey: "Qualitative Graphing Techniques", available at Mathematics Library and Mathematics Support Center. These have very good discussions of methods of graphing.

An Example with a Guide:

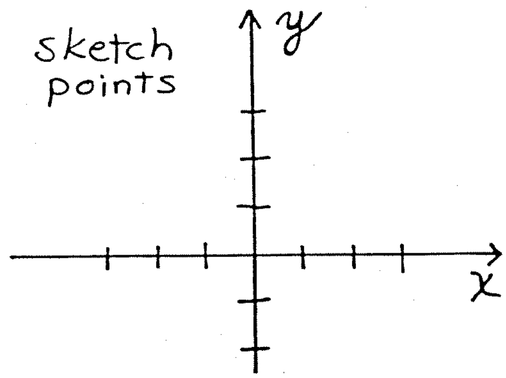
Sketch $\frac{1}{(x+2)} = -3y + 1$

(1) Is it familiar?

(2) Determine the intercepts:

$y = 0$

$x = 0$



(3) Is it symmetrical?

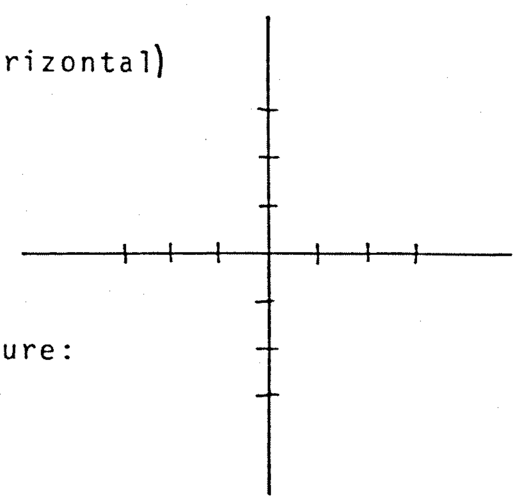
Try by plugging in $x = \pm 2$, then considering x and $-x$ in general.

(4) Are there asymptotes?

a) Set denominator = 0 (vertical asymptote)

b) What happens as $x \to \infty$, $x \to -\infty$. (Horizontal)

Add dotted lines to picture:



(5) Where is the curve?

Do you have any clues?

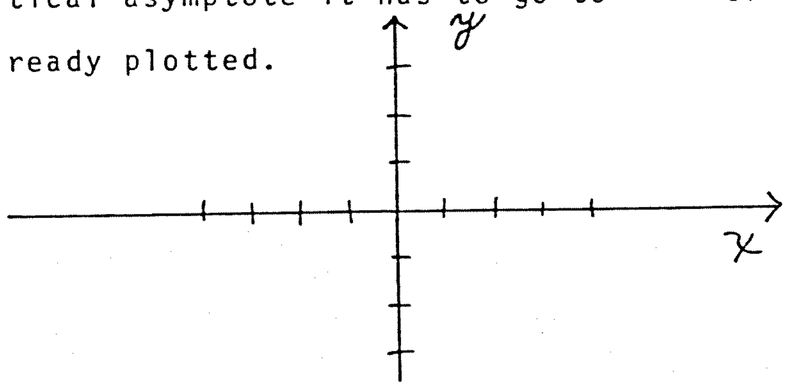
Hint: look at signs of $(x+2)$ and $(-3y+1)$

(6) Now can you sketch the curve?

Hint: it is a conic (and you know it's not a line, circle, or parabola)

As the curve approaches the vertical asymptote it has to go to $+\infty$ or $-\infty$ and you have some points already plotted.

Sketch the curve.



If you need more help in seeing what's happening near the asymptotes, plugging in numbers very close to them (the asymptotes) will give you an idea. Also see Hughes-Hallett, Elementary Functions, Chap. 4.2 for well discussed examples.

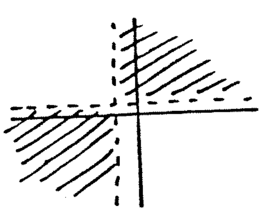
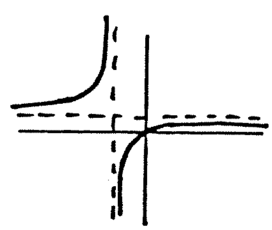
Solution:

1) It is a conic. $-3yx-6y+x+1 = 0$ is some conic that can't be put into a standard form. It isn't parabola (no x^2 term) or circle (no x^2+y^2 terms), or a line (not linear). Don't work with multiplied out form. (It's easier to leave it

2) $y = 0$, $x = -1$ and $x = 0$, $y = {}^1/6$

3) Not symmetrical.

4) Vertical asymptote: $x = -2$, Horizontal: $y = {}^1/3$

5)  hyperbola 

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Mathematics Support Capsules

GRAPHING

X. POST TEST

Questions:

1) Which of the following are symmetrical about the y-axis? ______________
Which are symmetric about the origin? ________________

a) $y = 1/x^2$

b) $y = (x+1)^2$

c) $y = -x$

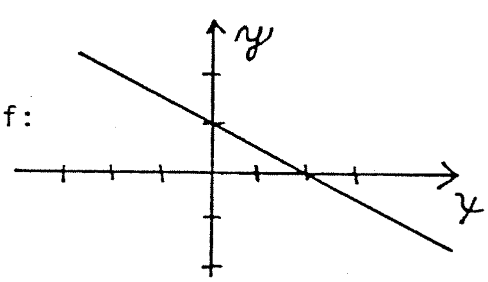
d) $y = x^6 + x^2$

e) $y = x^5 + x^3 + 2x$

2) Draw the graph of $x = 1/y^2$.

3) Sketch $f(x) = -(x-1)^3 + 3$ (DON'T PLOT POINTS!)

4) What is the equation of:



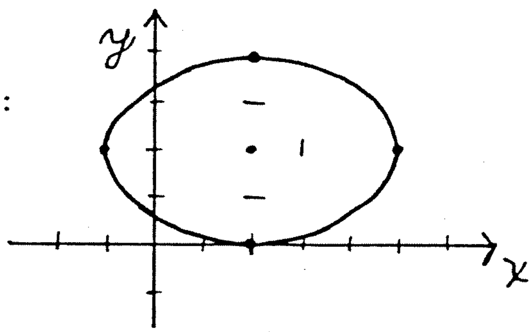
5) Sketch $y = -(x+4)^4(x-1)^3(x+1)$.

6) Find the horizontal and vertical asymptotes for:

a) $2y + 3 = \frac{1}{(x+1)(x-1)}$

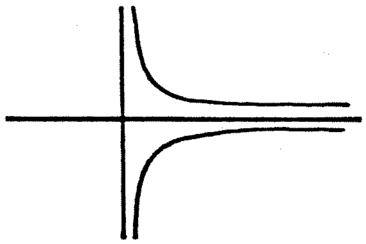
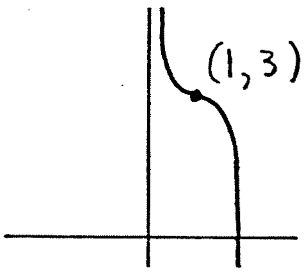
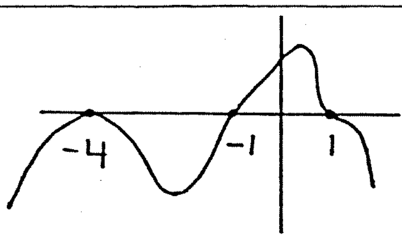
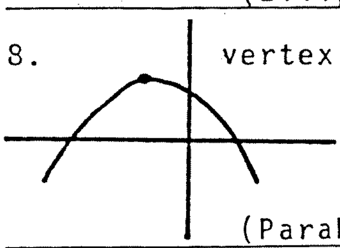
b) $x + 3 = 1/y$

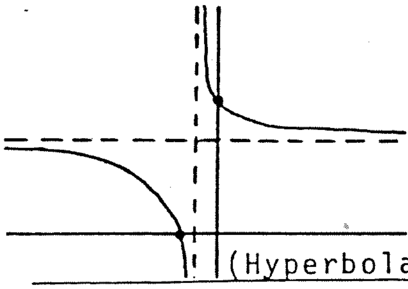
7) What is the equation for:



8) Graph: $6x + 6y + x^2 - 15 = 0$.

9) Graph: $2x + 1 = \frac{2}{3y-6}$

Answers to Post Test	Graphing Capsules to See	from D.Hughes Hallett* *Elementary Functions*
1. Symmetrical about y-axis <u>a,d</u> about origin <u>c,e</u>	II.Symmetry	Chap.3, sec.2
2. 	III.Graphs You Should Know	p.9
3. 	IV.New Graphs from Old	Chap.3, sec.3,4,5
4. $y = -2x+1$	V.Straight Lines	Chap.2
5. 	VI.Polynomials	Chap.4,Sec.1 See also Rickey*
6. a) $x = 1, x = -1,$ $y = -3/2$ b) $y = 0, x = -3$	VII.Finding Asymptotes	Chap.4,Sec.2 See also Rickey*
7. $\frac{(x-2)^2}{3^2} + \frac{(y-2)^2}{2^2} = 1$ (Ellipse)		
8. vertex(-3,4)  (Parabola)	VIII.Conics	Chap.5

Answers to Post Test	Graphing Capsules to See	from D.Hughes Hallett* Elementary Functions
9. Asymptotes: $y = x, x = -1/2$ Intercepts: $x = 0, y = 8/3$ $y = 0, x = -2/3$  (Hyperbola)	VIII.Conics	Chap.5

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