Current Topics in First Semester Calculus

(Exponentials, logarithms, integration by substitution)

Points to Watch on Exponentials and Logs

Basic Properties: (1) $a^x \cdot a^y = a^{x+y}$ (2) $(a^x)^y = a^{xy}$

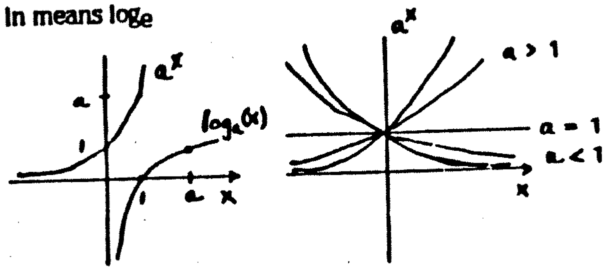
(3) $\log(x) + \log(y) = \log(xy)$ (4) $\log(x^y) = y \cdot \log(x)$

(5) $\log_a(x) = (\log(x))/\log(a)$

Notation: log means $\log_{10}$, or general log (doesn't matter what base)

ln means $\log_e$

Graphs:



Calculus Properties: $(d/dx)e^x = e^x$; note: if $y = e^x$, then $y' = y$

$(d/dx)\ln(x) = 1/x$ and $\int_1^x (1/t)(dt) = \ln(x)$

So $\int (1/(\text{cabin}))(d)(\text{cabin}) = \ln(\text{cabin}) + c$ = natural log cabin + c = houseboat

Importance: Imagine doing algebra without the manipulations possible thanks to: the additive identity (0); the multiplicative identity (1); substitution (y = x). Similarly, calculus needs e^x

We can use the above properties to simplify differentiation, e.g.

$y = \ln(x^5(2x^2 + 3)^7) \Rightarrow y' = ?$ (ugh!) BUT....

$y = 5\ln(x) + 7\ln(2x^2 + 3) \Rightarrow y' = (5/x) + ((7/(2x^2 + 3)) \cdot 4x$ (easy!)

Integration by Substitution

A Word about Integration: Integration is a fundamentally more difficult problem than differentiation. It is easy to check (by differentiation) but finding the right substitution or transformation involves "educated guessing" and a willingness to "try it and see." Many innocent-looking functions have no antiderivative formula. The classic example is e^{-x^2}. This means it is highly adventurous to write down an equation and try to integrate it. It may not be possible! But see the study tip (below) for an instructive way to generate problems that will work, together with their solutions.

Basic Idea: Substitute expressions to change an integral from one you don't know into one you do know!

General Pattern: Given $\int_{x=a}^{x=b} f(g(x)) \cdot g'(x)(dx)$, let $u = g(x)$, $du = g'(x)(dx)$

Then $\int f(u)(du) = F(u)\Big|_{x=a}^{x=b} = F(g(x))\Big|_a^b$, where F is the antiderivative of f

Examples: (a): $\int \cos(x)\sqrt{\sin(x)}(dx)$. Let $u = \sin(x)$, $du = \cos(x)(dx)$.

Then $\int \sqrt{u}(du) = \int u^{1/2}(du) = (2/3)u^{3/2} = (2/3)(\sin(x))^{3/2} + c$

(b): $\int x^3 e^{x^4}(dx) = (1/4)\int e^{x^4} \cdot 4x^3(dx)$. Let $u = x^4$, $du = 4x^3(dx)$

Then $(1/4)\int e^{x^4} \cdot 4x^3(dx) = (1/4)\int e^u(du) = (1/4)e^u = (1/4)e^{x^4} + c$

Study tip: Write down a function which requires the chain rule to differentiate it. Differentiate it, simplify, and then try to find the antiderivative. It will require a substitution.

e.g. $y = (\sin(x))^3 \Rightarrow y' = 3\sin^2(x)\cos(x)$. Now consider $\int 3\sin^2(x)\cos(x)(dx)$ & let $u = \sin(x)$, $du = \cos(x)(dx)$. So $\int 3u^2(du) = 3u^3/3 = u^3 = (\sin(x))^3 + c$

Careful study of examples you make up like this will make you an expert at spotting substitutions that will work!

Workshop Problems

(1) $\int 2\sin(x)\cos(x)(dx)$ (2) $\int ((\tan 3x)/(\cos^2 x))(dx)$

(3) $\int_{-1}^{1} (x^3+1)^3 \cdot 3x^2(dx)$ (4) $\int ((3x+1)/(3x^2+2x+1)^5)(dx)$

(5) $\int_{-2\pi}^{8\pi} \sin(\theta/2)(d\theta)$ (6) $\int (1/5)\ln 5(dx)$

(7) $\int \sqrt{4\sin(x)+\cos(x)}\cdot(-2\cos(x)+\sin(x)/2)(dx)$

(8) $\int xe^{-x^2}(dx)$ (9) $\int x^2/\sqrt{1+x}^3(dx)$

(10) $\int dt/\cos^2(2t)$ (11) $\int (x\cdot\sqrt[3]{1-x})(dx)$

Find inverse functions for the following (if it exists):

(1) $f(x)=x^3-1$ (2) $f(x)=1/(x^2+1)$ (3) $f(x)=1/(x-1)$

(4) $f(x)=\ln(x)$ (5) $f(x)=\sin(x)$ (6) $f(x)=\ln(5x)$ (7) $f(x)=e^{-x}+1$

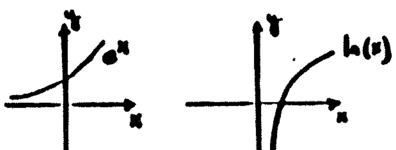
Find derivatives for the following:

(1) $f(x)=e^x$ (2) $f(x)=e^{kx}$ (3) $f(x)=e^{(\ln 10)x}$

(4) $f(x)=e^{\ln(10^x)}$ (5) $f(x)=e^{x^2}$ (6) $f(x)=\ln x$ (7) $f(x)=\ln(\tan(x))$

(8) $f(x)=\ln(x^2+1)$ (9) $f(x)=\ln(x/(1+x^2))$

Use these graphs to sketch the following:



(1) $f(x)=e^{-x}+1$ (2) $f(x)=\ln(1/x)$ (3) $f(x)=\ln(-x)$ (4) $f(x)=\ln(2x)$

<u>Word Problem #1</u>: A population grows at the rate of 5% per year. How many years does it take for the population to double?

<u>Word Problem #2</u>: The half life of plutonium is 24000 years. Write an expression for the amount of plutonium remaining from an initial 10 kg mass.

<u>Supplementary Problems</u> (Solve before the final!)

(1) $\int e^{\sin(x)}\cos(x)(dx)$ (2) $\int e^{\tan(x)}(1+\tan^2 x)(dx)$ (3) $(d/dx)(10^x)$

(4) $\int (e^{\tan x})/\cos^2 x(dx)$ (5) $\int \sin(x)/\cos(x)(dx)$ (6) $(d/dx)(x^x)$

(7) $(d/dx)(e^{\sin^2(x)})$ (8) $(d/dx)e^{2\cdot\ln\sin(x)}$ (9) $\int xe^{-\sin^2 x}\cos(x)(dx)$

(10) $10^{kx}=e^x$ (find k) (11) $\ln(y)=c\cdot\log_{10}y$ (find c) (12) $(d/dx)\log_{10}x$

(13) $e^{\ln(\sin(x))}=?$ (14) $\ln(e^{\cos(x)})=?$ (15) $\ln(1+2a)=1+\ln(a)$ (find a)

ANSWERS TO WORKSHOP PROBLEMS

(1) $u=\sin(x), du=\cos(x)(dx) \Rightarrow \int 2\sin(x)\cos(x)(dx) = 2\int \sin(x)\cos(x)(dx)$
$= 2\int u(du) = 2\cdot u^2/2 = \sin^2 x + c$

(2) $u=\tan(x), du=\sec^2(x)(dx) = 1/\cos^2(x)(dx) \Rightarrow \int \tan^3(x)(dx)/\cos^2(x)$
$= \int u^3(du) = u^4/4 = (\tan^4 x/4) + c$

(3) $u=x^3+1, du=3x^2(dx) \Rightarrow \int_{-1}^{1}(x^3+1)^3\cdot 3x^2(dx) = \int_{x=-1}^{x=1} u^3(du) = (u^4/4)\Big|_{x=-1}^{x=1}$
$=(x^3+1)^4/4\Big|_{-1}^{1} = (2^4/4) - 0 = 4$

(4) $u=3x^2+2x+1, du=(6x+2)(dx) \Rightarrow \int((3x+1)/(3x^2+2x+1)^5)(dx) = (1/2)\int(1/u^5)(du)$
$=(1/2)\int u^{-5}(du) = (1/2)u^{-4}/-4 = -(1/8)(3x^2+2x+1)^{-4}+c$

(5) $\int_{-2\pi}^{2\pi}\sin(\theta/2)(d\theta) = 0$ by inspection....hmmmn...that's odd

(6) $\int(1/5)\ln 5(dx) = (\ln 5/5)\int dx = (\ln 5/5)\cdot x + c$

(7) $u=4\sin(x)+\cos(x), du=(4\cos(x)-\sin(x))(dx) \Rightarrow$
$\int\sqrt{4\sin(x)+\cos(x)}\cdot(-2\cos(x)+\sin(x)/2)(dx) = -\frac{1}{2}\int\sqrt{u}(du) = -\frac{1}{2}\int u^{1/2}(du)$
$= -\frac{1}{2}\cdot(2/3)\cdot u^{3/2} = -(1/3)(4\sin(x)+\cos(x))^{3/2}+c$

(8) $u=x^2, du=2x(dx) \Rightarrow \int xe^{-x^2}(dx) = (1/2)\int e^{-x^2}\cdot 2x(dx) = (1/2)\int e^{-u}(du)$
$= -(1/2)e^{-u} = -(1/2)e^{-x^2}+c$, OR $= 0$, by inspection, on a symmetric interval!

(9) $u=1+x^3, du=3x^2(dx) \Rightarrow \int x^2(dx)/\sqrt{1+x^3} = (1/3)\int(du)/u^{1/2} = (2/3)u^{1/2}$
$=(2/3)(1+x^3)^{1/2}+c$

(10) $u=2t, du=2(dt) \Rightarrow \int(dt)/\cos^2(2t) = (1/2)\int\sec^2 u(du) = (1/2)\tan(u) = (1/2)\tan(2t)+c$

(11) $u=1-x \Rightarrow x=1-u, du=-(dx) \Rightarrow \int x\sqrt[3]{1-x}(dx) = -\int(1-u)u^{1/3}(du)$
$=-(3/4)u^{4/3}+(3/7)u^{7/3} = -(3/4)(1-x)^{4/3}+(3/7)(1-x)^{7/3}+c$

(1) $f^{-1}(x)$ exists: $x=y^3-1 \Rightarrow y=\sqrt[3]{x+1} = f^{-1}(x)$ (2) $f^{-1}(x)$ does not exist

(3) $f^{-1}(x)$ exists: $x=1/(y-1) \Rightarrow xy-x=1 \Rightarrow y=(1+x)/x=f^{-1}(x)$

(4) $f^{-1}(x)=e^x$ (5) $f^{-1}(x)$ does not exist (6) $f^{-1}(x)$ exists; $x=\ln 5y \Rightarrow$
$e^x=5y \Rightarrow y=(1/5)e^x=f^{-1}(x)$ (7) $f^{-1}(x)$ exists; $x=e^{-y}+1 \Rightarrow$
$x-1=e^{-y} \Rightarrow \ln(x-1)=-y \Rightarrow y=-\ln(x-1)=f^{-1}(x)$

(1) e^x (2) ke^{kx} (3) $\ln 10\cdot e^{(\ln 10)x}$ (4) $\ln 10\cdot e^{(\ln 10)x}$ (5) $e^{x^2}\cdot 2x$ (6) $1/x$
(7) $(1/\tan(x))\cdot\sec^2(x)$ (8) $(1/(x^2+1))\cdot 2x$

(9) $f(x)=\ln(x)-\ln(1+x^2) \Rightarrow f'(x)=(1/x)-(2x)/(1+x^2)$

WP#1: $1.05^t=2 \Rightarrow t\cdot\ln(1.05)=\ln 2 \Rightarrow t=(\ln 2)/\ln(1.05)$ **WP#2:** $M=10(1/2)^{t/24000}$

(1) $u=\sin(x), du=\cos(x)(dx) \Rightarrow \int e^{\sin(x)}\cos(x)(dx) = \int e^u(du) = e^u = e^{\sin(x)}+c$

(2) $u=\tan(x), du=\sec^2(x)(dx) = (1+\tan^2(x))(dx) \Rightarrow \int e^{\tan(x)}(1+\tan^2(x))(dx)$
$=\int e^u(du) = e^u = e^{\tan(x)}+c$

(3) $(\ln 10)(10^x)$ (4) $u=\tan(x), du=\sec^2 x(dx) = 1/\cos^2 x(dx) \Rightarrow \int(e^{\tan(x)})/\cos^2 x(dx)$
$=\int e^u(du) = e^u = e^{\tan(x)}+c$

(5) $u=\cos(x), du=-\sin(x)(dx) \Rightarrow \int(\sin(x)/\cos(x))(dx) = -\int(1/u)(du) = -\ln(u) = -\ln\cos(x)$
$=\ln(1/\cos(x)) = \ln(\sec(x))+c$ note $\sin(x)/\cos(x)=\tan(x)$, so this is $\int\tan(x)(dx)$

(6) $(d/dx)(x^x) = (d/dx)(e^{x\ln x}) = e^{x\ln x}(1\cdot\ln x + x\cdot(1/x)) = x^x(1+\ln x) = x^x + (\ln x)\cdot x^x$

(7) $e^{\sin^2 x}\cdot 2\sin(x)\cdot\cos(x)$ (8) $e^{2\ln\sin(x)} = e^{\ln\sin^2 x} = \sin^2 x$, so $f'(x) = 2\sin(x)\cos(x)$

(9) $u=\sin(x), du=\cos(x)(dx) \Rightarrow fcn = \pi\int e^{-u^2}(du) \Rightarrow$ impossible!

(10) $kx\ln 10 = x\ln(e) \Rightarrow k = (\ln(e))/\ln 10 = 1/\ln 10$

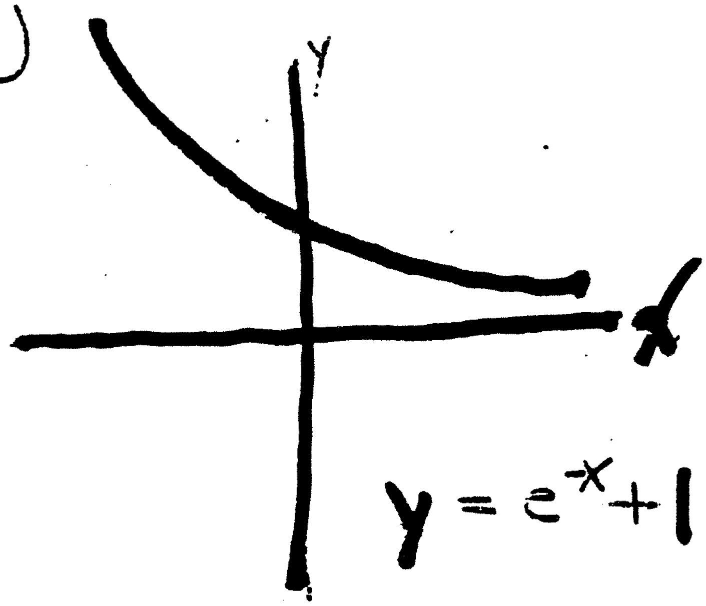
(11) Let $y=e^x$ from above. $c\cdot kx = x \Rightarrow c\cdot k = 1$. $k=1/\ln 10$ (from above) so $c=\ln(10)$

(12) $\log_{10}x = \ln(x)/\ln(10) \Rightarrow (d/dx)\log_{10}x = 1/(x\cdot\ln(10))$ (13) $\sin(x)$

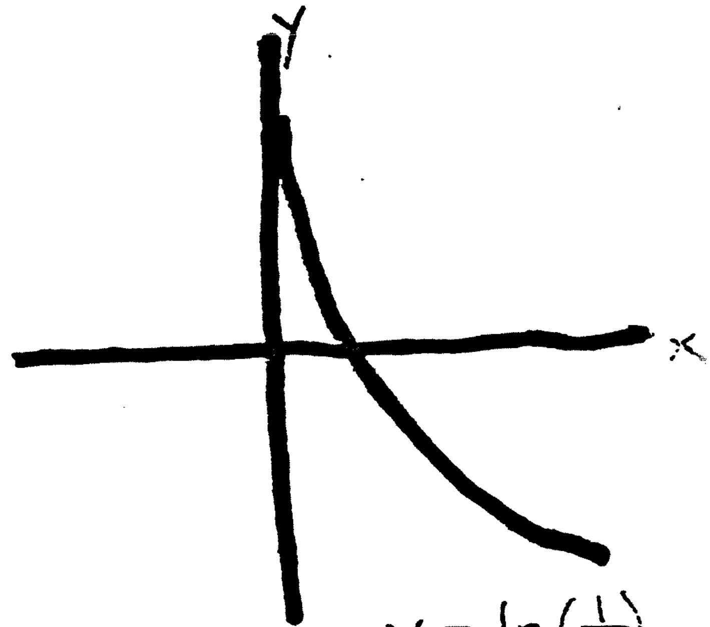
(14) $\cos(x)$ (15) $e^{\ln(1+2a)} = e^{1+\ln(a)} \Rightarrow 1+2a = e\cdot e^{\ln(a)} \Rightarrow 1+2a = ae \Rightarrow a = 1/(e-2)$

Solutions to Graphing Problems
(Exponentials, logs workshop)

1)

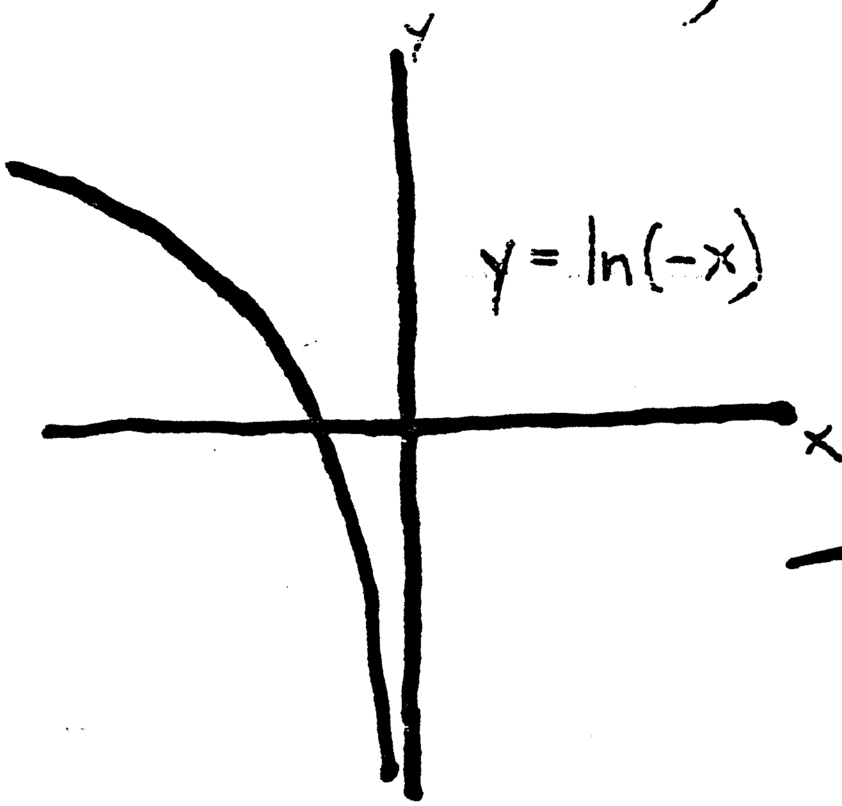


$y = e^{-x} + 1$



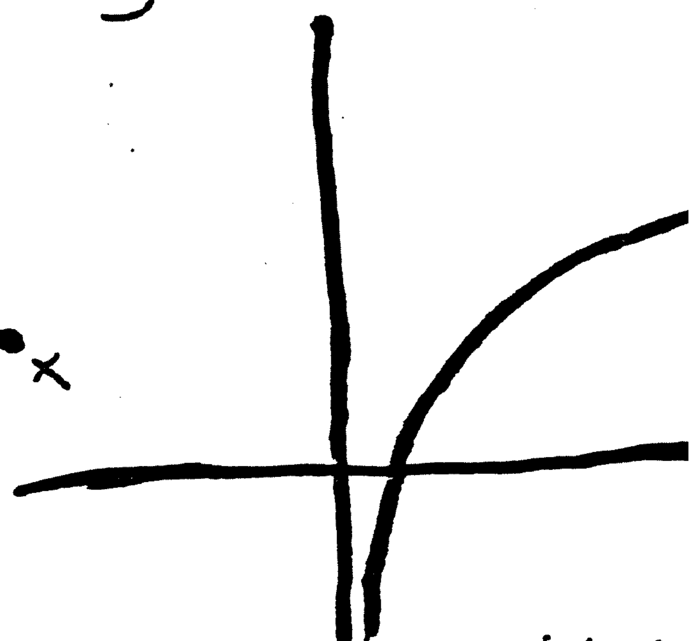
$y = \ln\left(\frac{1}{x}\right)$

3)



$y = \ln(-x)$

4)



$y = \ln(2x)$