FAMOUS SERIES

a) Geometric Series $1 + x + x^2 + x^3 + \dots$ (converges if $|x| < 1$) (diverges if $|x| > 1$)
NOTE: sum is $1/(1 - x)$ for $|x| < 1$

b) P - Series $\sum \frac{1}{n^p}$ (diverges for $p \le 1$)

c) Harmonic Series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ (diverges)

d) Alternating Harmonic Series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$ (converges)

e) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

f) $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

g) $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

h) $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

i) $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

<u>Notes:</u>

e,f,g: $e^x = \cos(ix) - i\sin(ix)$ where $i = \sqrt{-1}$

h,d: x=1 in h) yields d) so the sum of the Alt. Harm. Series is $\ln(1+1) = \ln 2$

h,c: informally, $x = -1$ makes h) blow up ($\ln 0$) in accordance with what we know about c)

b,c: c) is a special case of b) with $p = 1$

a: sum of a) is $1/(1 - x)$ for $|x| < 1$

h,a: h) may be found by integrating a) with $x=-t$: $\int \frac{1}{1-(-t)}\,dt = \int \frac{1}{1+t}\,dt = \int 1 - t + t^2 - t^3 + \dots dt$

i,a: i) may be found by integrating a) with $x=-t^2$: $\tan^{-1}x = \int \frac{1}{1+t^2}\,dt = \int 1 - t^2 + t^4 - t^6 + t^8 - \dots dt$

Useful Limits

a) $\lim_{h\to 0} (1+h)^{1/h} = e$

b) $\lim_{n\to\infty} (1 + \frac{1}{n})^n = e$

c) $\lim_{n\to\infty} (1 - \frac{1}{n})^n = e^{-1}$ note: c) is a special case of d) with $x = -1$

d) $\lim_{n\to\infty} (1 + \frac{x}{n})^n = e^x$

e) $\lim_{n\to\infty} \frac{1}{n^p} = 0$ if $p > 0$

f) $\lim_{n\to\infty} x^n \begin{cases} = 0 \text{ if } |x| < 1 \\ \text{diverges if } |x| > 1 \end{cases}$

g) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for <u>all</u> x

h) $\lim_{n\to\infty} x^{1/n} = 1$ for $x > 0$

i) $\lim_{n\to\infty} n^{1/n} = 1$

j) $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ (limit of a sum is the sum of the limits)

k) $\lim_{n\to\infty} (a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$ (limit of a product is the product of the limits)

l) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ (denom. $\neq 0$) (limit of a quotient is the quotient of the limits)

m) "Flyswatter Theorem": If $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

then $\lim_{n\to\infty} b_n = L$

n) ratios of polynomials tend to ($\lim x\to\infty$) $\begin{cases} \text{the ratio of leading coefficients if of same degree} \\ 0 \quad \text{if degree of denominator larger} \\ \text{diverge} \quad \text{if degree of numerator larger} \end{cases}$

Useful Inequalities

a) $|\sin(x)| \le 1$

b) $|\csc(x)| \ge 1$

c) $|\cos(x)| \le 1$

d) $|\sec(x)| \ge 1$

e) $|\sin(x)| \le |\tan(x)|$

f) $n^2 + 1 \le (n+1)^2$

"eventually" g) $\ln n > 1$

h) $\ln n < n^k$ for any $k > 0$

i) $n > 1$ (!) Don't forget this when using TD

- Often the hardest part of showing convergence or divergence of a series is the indecision: What do I believe it does? After all, if you try to show a series converges when it actually diverges, you'll have difficulty!
- The limits of the last section can help a lot with the Test for Divergence. Together with inequalities you can often get an idea of what to try to show. If the individual terms of the series "look like" n^3/n^4 as $n \to \infty$, then the series "looks like" $1/n$ and so you want to show it diverges.
- Many limits boil down to "look like" ratios of polynomials after stripping out trig functions using the inequalities above.
- h) leads to the peculiar rule of thumb that in ratios $\ln n$ "looks like" 1 since any positive power of n will dominate it. For example, when you see

$\sum \frac{\ln n}{n^2}$, <u>think</u> $\sum \frac{1}{n^2}$ to see that it converges.

Show it <u>carefully</u> by using $\ln n < \sqrt{n}$, "eventually"

So $\sum \frac{\ln n}{n^2} < \sum \frac{\sqrt{n}}{n^2} = \sum \frac{1}{n^{3/2}}$, a convergent p-series

New Series From Old

If you have a series expression, you can instantly create new, interesting series using all the techniques you have to create new functions from old familiar ones.

<u>Multiply it by a constant</u>

Before: $1 + x + x^2 + x^3 + \dots \quad = 1/(1-x)$

After: $a + ax + ax^2 + ax^3 + \dots$ (mult. by a) $\quad = a(1/(1-x)) = a/(1-x)$

<u>Substitute an expression for x:</u> (e.g. let $x = -t^2$)

Before: $1 + x + x^2 + x^3 + \dots \quad = 1/(1-x)$

After: $1 - t^2 + t^4 - t^6 + \dots \quad = 1/(1+t^2)$

<u>Multiply by a power of x:</u> (e.g. x^2)

Before: $1 + x + x^2 + x^3 + \dots \quad = 1/(1-x)$

After: $x^2 + x^3 + x^4 + x^5 + \dots \quad = x^2/(1-x)$

<u>Integrate an Expression:</u>

Before: $1 - x + x^2 - x^3 + x^4 - \dots \quad = 1/(1+x)$

After: $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad = \int \frac{1}{1+x}\,dx = \ln|1+x|$

<u>Differentiate an Expression:</u>

Before: $1 + x + x^2 + x^3 + \dots \quad = 1/(1-x)$

After: $0 + 1 + 2x + 3x^2 + 4x^3 + \dots \quad = \frac{d}{dx} \cdot \frac{1}{1-x} = (1-x)^{-2} = \frac{-1}{(1-x)^2}$

Review of Convergence Tests

NAME	STATEMENT	COMMENTS
Divergence test	If $\lim_{k\to+\infty} u_k \neq 0$, then $\sum u_k$ diverges.	If $\lim_{k\to+\infty} u_k = 0$, $\sum u_k$ may or may not converge.
Integral test	Let $\sum u_k$ be a series with positive terms and let $f(x)$ be the function that results when k is replaced by x in the formula for u_k. If f is decreasing and continuous for $x \geq 1$, then $\sum_{k=1}^{\infty} u_k$ and $\int_1^{+\infty} f(x)\,dx$ both converge or both diverge.	Use this test when $f(x)$ is easy to integrate.
Comparison test	Let $\sum a_k$ and $\sum b_k$ be series with positive terms such that $a_1 \leq b_1, a_2 \leq b_2, \ldots, a_k \leq b_k, \ldots$ If $\sum b_k$ converges, then $\sum a_k$ converges; and if $\sum a_k$ diverges, then $\sum b_k$ diverges.	Use this test as a last resort. Other tests are often easier to apply.
Ratio test	Let $\sum u_k$ be a series with positive terms and suppose $\lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \rho$ (a) Series converges if $\rho < 1$. (b) Series diverges if $\rho > 1$ or $\rho = +\infty$. (c) No conclusion if $\rho = 1$.	Try this test when a_k involves factorials or kth powers.

Review of Convergence Tests (Continued)

NAME	STATEMENT	COMMENTS
Root test	Let $\sum u_k$ be a series with positive terms such that $\rho = \lim_{k\to+\infty} \sqrt[k]{u_k}$ (a) Series converges if $\rho < 1$. (b) Series diverges if $\rho > 1$ or $\rho = +\infty$. (c) No conclusion if $\rho = 1$.	Try this test when u_k involves kth powers.
Limit comparison test	Let $\sum a_k$ and $\sum b_k$ be series with positive terms such that $\rho = \lim_{k\to+\infty} \frac{a_k}{b_k}$ If $0 < \rho < +\infty$, then both series converge or both diverge.	This is easier to apply than the comparison test, but still requires some skill in choosing the series $\sum b_k$ for comparison.
Alternating series test	The series $a_1 - a_2 + a_3 - a_4 + \cdots$ and $-a_1 + a_2 - a_3 + a_4 - \cdots$ converge if (a) $a_1 \geq a_2 \geq a_3 \geq \cdots$ (b) $\lim_{k\to+\infty} a_k = 0$	This test applies only to alternating series.
Ratio test for absolute convergence	Let $\sum u_k$ be a series with nonzero terms such that $\rho = \lim_{k\to+\infty} \frac{\lvert u_{k+1}\rvert}{\lvert u_k\rvert}$ (a) Series converges absolutely if $\rho < 1$. (b) Series diverges if $\rho > 1$ or $\rho = +\infty$. (c) No conclusion if $\rho = 1$.	The series need not have positive terms and need not be alternating to use this test.

Note:

Still need telescoping series,

"p" series : $\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \mathbb{C} \text{ if } p > 1 \\ \mathbb{D} \text{ if } p \leq 1 \end{cases}$

& geometric series: $\sum_{n=1}^{\infty} x^{n-1} \begin{cases} \mathbb{C} \text{ to } \frac{1}{1-x} \ \forall \lvert x\rvert < 1 \\ \mathbb{D} \ \lvert x\rvert \geq 1 \end{cases}$

Infinite Series Workshop Fact Sheet

Geometric Series: With c a constant, $\sum_{n=0}^{\infty} cx^n = c/(1-x)$ if and only if $|x| < 1$. Otherwise, it diverges.

p-series: With c a constant, $\sum c/n^p$ diverges for $p \leq 1$.

Test for Divergence: (TD) $\sum a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$. Another way of saying this is: If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. It does not say that if $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.

Integral Test: (IT) If f is a continuous function that is positive and decreasing for $x \geq 1$, and $f(n) = a_n$ (for $n = 1,2,3,\ldots$), then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Leftrightarrow \int_1^{\infty} f(x)dx \text{ converges}$$

Basic Comparison Test: (BCT) Let $\sum a_n$ and $\sum b_n$ be two series such that $0 \leq a_n \leq b_n$, for large n. Then $\sum b_n$ converges $\Rightarrow \sum a_n$ converges, and $\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges.

Limit Comparison Test: (LCT) Suppose $\sum a_n$ and $\sum b_n$ are positive series and that $\lim_{n\to\infty}(a_n/b_n) = L$.

1. If $L > 0$ (and $L \neq \infty$), then if either series converges (diverges), then so does the other (i.e. they do the same thing).
2. If $L = 0$, then if $\sum b_n$ converges, then so does $\sum a_n$; if $\sum a_n$ diverges, so does $\sum b_n$.
3. If $L = \infty$, then if $\sum a_n$ converges, then so does $\sum b_n$; if $\sum b_n$ diverges, so does $\sum a_n$.

Ratio Test: (RAT) For a series $\sum a_n$, let $\rho = \lim_{n\to\infty}(|a_{n+1}|/|a_n|)$; then

$\rho < 1 \Rightarrow \sum a_n$ converges absolutely
$\rho > 1 \Rightarrow \sum a_n$ diverges
$\rho = 1 \Rightarrow$ test fails to distinguish

Root Test: (ROOT) For a series $\sum a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$; then

$\rho < 1 \Rightarrow \sum a_n$ converges absolutely
$\rho > 1 \Rightarrow \sum a_n$ diverges
$\rho = 1 \Rightarrow$ test fails

Alternating Series Test: (AST) $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges if the sequence $a_1, a_2, a_3, \ldots$ decreases to the limit 0.

Series Workshop Practice Sheet
(Covers Gillett 12.2 - 12.6)

Part I: Find the sum of the following series, if they converge:

1. $\sum_{n=0}^{\infty} e^{-n}$
2. $\sum_{n=2}^{\infty} (\pi/3)^n$
3. $\sum_{n=1}^{\infty} \ln(n/n+1)$
4. $\sum_{n=1}^{\infty} (2/3)^n$
5. $\sum_{n=1}^{\infty} 2/(2n+1)(2n+3)$

Part II: Determine whether the following infinite series converge or diverge and state which test you are using.

1. $\sum 4/2^n$
2. $\sum (1+1/n)^n$
3. $\sum 5^{2n}/n!$
4. $\sum \cos^2 n/n^3$
5. $\sum 1/\sqrt{n^2-2}$
6. $\sum 3/n^{2/3}$
7. $\sum n^2/2^n$
8. $\sum ne^{-n}$
9. $\sum 2/(\sqrt{n}+1)$
10. $\sum_{n=1}^{\infty} \cos(\pi n)/n$
11. $\sum n/\ln(n)$
12. $\sum 1/\sqrt{n^2+2}$
13. $\sum_{n=2}^{\infty} (-1)^n/\sqrt{n^2-1}$
14. $\sum (\sin^4 n/3)^n$
15. $\sum \ln(n)/n$
16. $\sum 1/\sqrt[3]{n^4-2}$
17. $\sum (3n^4+5n^3+2)/(n^2-1)(n^2+1)$
18. $\sum_{n=1}^{\infty} (1/2+1/n)^n$

Part III: Find all values of x for which the following series converge:

1. $\sum (x-2)^n/n^2$
2. $\sum x^{2n-1}/(2n-1)!$

Solutions to Series Practice Sheet

Part I:

1. $\sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} (1/e)^n = 1/(1-1/e) = e/(e-1)$ (geometric series)

2. $\sum_{n=1}^{\infty} (\pi/3)^n$; $|\pi/3| > 1$, so series is divergent geometric series.

3. $\sum_{n=1}^{\infty} \ln(n/n+1) = \sum_{n=1}^{\infty} \ln(n) - \ln(n+1)$, so
$S_n = 0 - \ln 2 + \ln 2 - \ln 3 + \ln 3 - \ln 4 + \ldots + \ln(n) - \ln(n+1)$.
Thus $\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\ln(n+1) = -\infty$, so series diverges.

4. $\sum_{n=1}^{\infty} (2/3)^n = -1 + \sum_{n=0}^{\infty} (2/3)^n = -1 + (1/(1-(2/3))) = -1+3 = 2$.
($\sum_{n=0}^{\infty} (2/3)^n$ is a geometric series with $|2/3| < 1$).

5. $\sum_{n=1}^{\infty} 2/(2n+1)(2n+3) = \sum_{n=1}^{\infty} (1/(2n+1))-(1/(2n+3))$ (partial fractions), so
$S_n = (1/3) - (1/5) + (1/5) - (1/7) + (1/7) - \ldots + (1/(2n+1)) - (1/(2n+3)) = (1/3)-(1/(2n+3))$. So $\sum_{n=1}^{\infty} 2/(2n+1)(2n+3) = \lim_{n\to\infty} S_n = 1/3$.

Part II: (There are often several ways to do these, but there is always only one answer.)

1. $\sum 4/2^n = \sum 4(1/2)^n$, a convergent geometric series.

2. $\sum (1+1/n)^n$ diverges by TD, since $\lim_{n\to\infty} (1+1/n)^n > 1 \neq 0$ [NOTE: THIS DOES $= e$]

3. $\sum 5^{2n}/n!$ converges by RAT.

4. $\sum \cos^2 n/n^3$ converges by BCT; $(\cos^2 n/n^3) \le (1/n^3)$ and $\sum (1/n^3)$ converges (p-series, $p > 1$)

5. $\sum 1/\sqrt{n^2-2}$ diverges by BCT; $(1/\sqrt{n^2-2}) \ge (1/n)$ and $\sum (1/n)$ diverges. (harmonic series)

6. $\sum 3/n^{2/3}$ is a divergent p-series, $p = (2/3) < 1$.

7. $\sum n^2/2^n$ converges by RAT.

8. $\sum ne^{-n}$ converges by IT. $\int_1^{\infty} xe^{-x}dx$ (use parts: $u=x$, $du=dx$; $dv = e^{-x}dx$, $v = -e^{-x}$) $= \lim_{b\to\infty} -xe^{-x}\Big|_1^b + \int_1^b e^{-x}dx = \lim_{b\to\infty} -xe^{-x} - e^{-x}\Big|_1^b =$
$\lim_{b\to\infty} (-b/e^b)-(1/e^b)-(-1/e-1/e) = 2/e$, so $\int_1^{\infty} xe^{-x}dx$ converges.

9. $\sum 2/(\sqrt{n}+1)$ diverges by LCT: $\lim_{n\to\infty} (1/\sqrt{n})/(2/\sqrt{n}+1) =$
$\lim_{n\to\infty} (\sqrt{n}+1)/2\sqrt{n} = \lim_{n\to\infty} (1+1/\sqrt{n})/2 = 1/2 = L$. (part 1 of test applies). Since $\sum 1/\sqrt{n} = \sum 1/n^{1/2}$ is a divergent p-series.

10. $\sum_{n=1}^{\infty} \cos(\pi n)/n$ converges by AST.

11. $\sum n/\ln(n)$ diverges by TD: $\lim_{n\to\infty} n/\ln(n) = \lim_{n\to\infty} 1/(1/n) = \lim_{n\to\infty} n = \infty \neq 0$.

12. $\sum 1/\sqrt{n^2+2}$ diverges by LCT: $\lim_{n\to\infty} (1/n)/(1/\sqrt{n^2+2}) = \lim_{n\to\infty} \sqrt{n^2+2}/n =$
$\lim_{n\to\infty} \sqrt{1+2/n^2}/1 = 1 = L$. (part 1 of test applies). $\sum 1/n$ is divergent (harmonic series).

13. $\sum_{n=2}^{\infty} (-1)^n/\sqrt{n^2-1}$ converges by AST.

14. $\sum (\sin^4 n/3)^n$ converges by BCT: $(\sin^4 n/3)^n \le (1/3)^n$ and $\sum (1/3)^n$ converges (geometric series $|1/3| < 1$)

15. $\sum \ln(n)/n$ diverges by IT: $\int_1^{\infty} (\ln x/x)(dx)$ (with $u = \ln x$, $du = dx/x$) $=$
$\lim_{b\to\infty} \int_0^b u\,du = \lim_{b\to\infty} (1/2)u^2\Big|_0^b = \lim_{b\to\infty} (1/2)b^2 = \infty$, so $\int_1^{\infty} (\ln x/x)(dx)$ diverges.

16. $\sum 1/\sqrt[3]{n^4-2}$ converges by LCT: $\lim_{n\to\infty} (1/n^{4/3})/(1/\sqrt[3]{n^4-2}) = \lim_{n\to\infty} \sqrt[3]{n^4-2}/n^{4/3}$
$= \lim_{n\to\infty} (\sqrt[3]{1-2/n^4})/1 = 1 = L$. (Part 1 of test applies)

17. $\sum (3n^4+5n^3+2)/(n^2-1)(n^2+1) = \sum (3n^4+5n^3+2)/(n^4-1)$ diverges by TD:
$\lim_{n\to\infty} (3n^4+5n^3+2)/(n^4-1) = \lim_{n\to\infty} (3+5/n+2/n^4)/(1-1/n^4) = 3 \neq 0$.

18. $\sum_{n=1}^{\infty} (1/2+1/n)^n$ converges by ROOT.

Part III:

1. $\sum (x-2)^n/n^2$; $\rho = \lim_{n\to\infty} ((|x-2|)^{n+1}/(n+1)^2) \times (n^2/|x-2|^n)$ goes to $|x-2|$ as n approaches ∞ $\Rightarrow$ absolute convergence for $1 < x < 3$. Series also converges absolutely at endpoints $x=1$ and $x=3$.

2. $\sum x^{2n-1}/(2n-1)!$; $\rho = \lim_{n\to\infty} (|x|^{2n+1})/(2n+1)! \times (2n-1)!/|x|^{2n-1}$ goes to 0 as n approaches ∞ $\Rightarrow$ absolute convergence for all x.